RATIONAL CURVES ON V_5 AND RATIONAL SIMPLE CONNECTEDNESS

ANDREA FANELLI, LAURENT GRUSON, AND NICOLAS PERRIN

Abstract. In this paper study rationality properties of genus zero stable maps on the quintic Fano threefold $V_5 \subset \mathbb{P}^6$. We prove the unirationality of the moduli spaces $\overline{M}^{\mathrm{bir}}_{0,0}(V_5, d)$ and that V_5 is strongly rationally simply connected.

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Introduction

Given a smooth polarised rationally connected variety X over $\mathbb{C}$, we are interested in studying the Kontsevich moduli spaces $\overline{M}_{0,m}(X, d)$ of rational curves on X with sufficiently large degree d. Our main motivation comes from the series of works by de Jong and Starr (cf. [dJS06b], [dJS06c], [dJS06a], [dJS07], etc.). Starting form the seminal work [GHS03], whose main result guarantees that a rationally connected fibration over a smooth curve has a section, de Jong and Starr explore numerical and geometric conditions on the general fibre of a rationally connected fibration over a surface which guarantee the existence of a rational section.

This deep analysis originated the new notion of *rational simple connectedness* and was applied to the case of homogeneous varieties to prove *Serre's Conjecture II* (cf. [dJHS11]). This notion has several variations, but heuristically is the algebraic analogue of simple connectedness in topology. In this paper we will use the following simplified definition focusing of the case of Fano varieties of Picard number one.

Let $\overline{M}_{0,m}(X, \beta)$ denote the (coarse) moduli space of stable m-pointed rational curves on X of degree $\beta \in \mathbb{Z}$. Let $\mathrm{ev}\colon \overline{M}_{0,m}(X, \beta) \to X^m$ denote the evaluation morphism. We focus our attention on the following moduli subspace in $\overline{M}_{0,m}(X, \beta)$. Let $\overline{M}^{\mathrm{bir}}_{0,m}(X, \beta)$ denote the closure in $\overline{M}_{0,m}(X, \beta)$ of the moduli space of morphisms which are birational onto their image.

Date: June 13, 2020.

2010 *Mathematics Subject Classification.* 14J45, 14H10, 14E05, 14E20.

The space $\overline{M}_{0,0}^{\mathrm{bir}}(X,\beta)$ is not irreducible in general Our definition of rational simple connectedness will require some irreducibility of $\overline{M}_{0,0}^{\mathrm{bir}}(X,\beta)$. Following the approach of [dJS06a], we provide a weak and a strong version of rationally simply connectedness.

Definition. *Let X be a Fano variety with Picard number 1.*

(1) *The variety X is* **rationally simply m-connected** *if there exists an integer $d_m > 0$ such that, for all $d \geq d_m$, the following holds:*
  - *the moduli space $\overline{M}_{0,0}^{\mathrm{bir}}(X,d) \subset \overline{M}_{0,0}(X,d)$ is irreducible;*
  - *the evaluation map $\mathrm{ev}_m \colon \overline{M}_{0,m}^{\mathrm{bir}}(X,d) \to X^m$ is dominant and its general fibre is rationally connected.*

(2) *The variety X is* **weakly rationally simply connected** *if X is rationally simply 2-connected.*

(3) *The variety X is* **strongly rationally simply connected** *if X is rationally simply m-connected for all $m \geq 2$.*

The reason why one requires this property for large enough d is that the behaviour of moduli spaces of rational curves in low degree can be atypical or, more simply, these spaces can be empty.

In [dJS06a], the authors also introduce a strong version of rational simple connectedness, which requires the existence of a "very free" ruled surface in X (a so called *very twisting scroll*). In this paper we start from the easier notion introduced in [dJS06a, Section 1] for varieties with Picard-rank one but whe shall implicitly use ideas coming from twisting scrolls in Section 2

Rational simple connectedness is subtle and very few examples are known; the picture has been clarified for complete intersections in projective spaces ([dJS06a], [DeL15]), homogeneous spaces ([dJHS11], [BCMP13]), and hyperplane sections of Grassmannians ([Fin10]). The first main result of this paper is the following.

Theorem A. (= Theorem 2.1) *The Fano threefold $V_5 \subset \mathbb{P}^6$, obtained as linear section of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$, is strongly rationally simply connected.*

The importance of this example comes from its very specific geometry: it is a smooth rational quasi-homogeneous (with respect to a SL_2-action) Fano threefold which is not *2-Fano* (cf. [AC12], [AC13]).

To prove this result, we adapt the *twisting-surface* technique of de Jong and Starr. One problem is that there is no twisting-scroll: twisting-surfaces rules by lines. Therefore we need to consider surfaces ruled by conics (see Section 2).

Although rational simple connectedness in *not* a birational property, the birational geometry of $V_5 \subset \mathbb{P}^6$ is well known (cf. Section 1) and we use it to developp another strategy for proving rationality results on moduli spaces of rational curves, reducing the study of moduli spaces of rational curves on V_5 to some special moduli spaces on the quadric threefold $\mathcal{Q}_3$ (cf. Section 3). This method gives unirationality results for the fibers of the evaluation map at less than two points and also proves the following unirationality result.

Theorem B. (= Theorem 3.1) *For any $d \geq 1$ the moduli space $\overline{M}_{0,0}^{\mathrm{bir}}(V_5,d)$ of stable maps birational onto their image is irreducible, unirational of dimension $2d$.*

This paper is organised as follows. In Section 1, we recall the definition of V_5 and its first properties especially the geometry of its lines and conics and the so called projection from a line (Lemma 1.3). Section 2 is devoted to the proof of the strong rational simple connectedness via twisting-surfaces ruled in conics techniques. In Section 3 we use Lemma 1.3 to prove unirationality results on the moduli space of stable maps. In the last Section 4, we conclude studying rationality of moduli spaces of rational curves on V_5 in low degree: new proofs of rationality of moduli spaces of quintic and sextic rational curves are provided, via explicit birational geometric methods.

Acknowledgement. We like to thank Sasha Kuznetsov, Andrea Petracci and Francesco Zucconi for interesting comments related to this work. The first-named author is currently funded by the Fondation Mathématique Jacques Hadamard.

For later use we fix some further notation.

Notation 0.1. Let X be a Fano variety of Picard number 1 and let $m \geq 0$ be an integer. Then we will often consider the fibre of the morphism

$$\Psi_m := \phi_m \times \mathrm{ev}_m \colon \overline{M}^{\mathrm{bir}}_{0,m}(X,d) \to \overline{M}_{0,m} \times X^m, \tag{0.A}$$

where $\phi_m \colon \overline{M}^{\mathrm{bir}}_{m}(X,d) \to \overline{M}_{0,m}$ is the natural morphism to the Deligne-Mumford moduli space of m-marked rational curves.
Let $\mathbf{t} = (t_1, \dots, t_m) \in (\mathbb{P}^1)^m$ be a marking on $\mathbb{P}^1$. The corresponding class in $\overline{M}_{0,m}$ will be also denoted by $\mathbf{t}$, to simplify the notation. Moreover, fix m points $\mathbf{x} = (x_1, \dots, x_m)$ of X. Assume $m \geq 3$. Then the fibre

$$\Psi_m^{-1}(\mathbf{t}, \mathbf{x}) \cong_{\mathrm{bir}} \mathrm{Mor}_d^{\mathbf{t} \to \mathbf{x}}(\mathbb{P}^1, X)$$

is birational to the variety of degree d morphisms $f \colon \mathbb{P}^1 \to X$ such that $f(\mathbf{t}) = \mathbf{x}$ which are birational onto the image. If $m \leq 2$, the previous relation holds modulo $\mathrm{Aut}(\mathbb{P}^1, \mathbf{t})$.

1. The Fano threefold V_5.

1.1. **Definition of V_5.** We introduce the central object of this section. For more details one can look at [IP99, pag. 60-61], [San14, Section 2], [CS16, Chapter 7] and [KPS18, Section 5.1].

Definition 1.1. A smooth Fano threefold X with $\rho(X) = 1$, Fano index $\iota(X) = 2$ and degree $(H^3) = 5$ is denoted by V_5.

From Iskovskikh's classification of smooth Fano threefolds with $\rho = 1$ (cf. [IP99, Section 12.2]) we know that V_5 is isomorphic to the linear section of the Grassmannian $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ by a general linear subspace $\mathbb{P}^6 \subset \mathbb{P}^9$.

Since V_5 verifies the index condition $\iota = n - 1$, where n is the dimension, in the literature, sometimes authors refer to V_5 as the *del Pezzo threefold* of degree 5.

Let us look now at the $\mathrm{SL}_2(\mathbb{C})$-action on V_5: one takes a vector space U with $\dim U = 2$, chooses a basis of $\wedge^2 U$ to identify U and its dual $U^\vee$. Let us denote by $S_n := \mathrm{Sym}^n(U)$ the symmetric tensor and consider the Clebsch-Gordan decomposition of $\wedge^2 S_4$ as $\mathrm{SL}(U)$-module: $\wedge^2 S_4 \simeq S_2 \oplus S_6$.

This shows how to induce a natural $\mathrm{SL}(U)$-action on $V_5 = \mathrm{Gr}(2, S_4) \cap \mathbb{P}S_6$, which is the intersection of two $\mathrm{SL}(U)$-invariant varieties.

Using the description provided in [MU83, Section 3], one deduces the orbit structure V_5 with respect to the $\mathrm{SL}_2(\mathbb{C})$-action (see also [IP99, pag. 60-61]).

Lemma 1.2. [MU83, Lemmas 1.5-1.6] *There is an action of* $\mathrm{SL}_2(\mathbb{C})$ *on* V_5 *with three orbits with the following description;*

- *a* 1*-dimensional orbit* σ *(with representative* $[x^6] \in \mathbb{P}S_6$*) which is a rational normal sextic in* $\mathbb{P}^6$*;*
- *a* 2*-dimensional orbit* $E \setminus \sigma$ *(with representative* $[x^5y] \in \mathbb{P}S_6$*), where* E *is a quadric surface which is the tangential scroll of* σ*. The normalisation* $\nu \colon \mathbb{P}^1 \times \mathbb{P}^1 \to E$ *is determined by a (non-complete) system of degree* $(1,5)$*;*
- *a* 3*-dimensional orbit* U *(with representative* $[xy(x^4+y^4)] \in \mathbb{P}S_6$*).*

The following construction is essential for our analysis of unirationality in Section 3. This can be found in [IP99, pag 147, Example (i)], [CS16, Section 7.7].

Lemma 1.3. *Let* l *be a line in* V_5*. Then the projection* $\phi_l \colon V_5 \dashrightarrow \mathbb{P}^4$ *from* l *is dominant on a quadric threefold* $\mathcal{Q}_3$ *and is birational. In particular* V_5 *is a rational threefold.*

Let D_l *be the divisor spanned by the lines in* V_5 *meeting* l*. Then* D_l *is a hyperplane section of* $V_5 \subset \mathbb{P}^6$ *and* $\phi_l(D_l) = \gamma_l$ *is a twisted cubic in* $\mathcal{Q}_3$*.*

1.2. **Lines, conics and first results on rational curves on V_5.** Let us list some remarkable properties which hold for some moduli spaces of rational curves on V_5. For more details, see [KPS18, Section 5.1].

The moduli space $\overline{M}_{0,0}(V_5,1)$ is isomorphic to the Hilbert scheme of lines and (cf. [Isk79, Proposition 1.6(i)]) $F_1(V_5) := \mathcal{H}_{0,1} \simeq \mathbb{P}S_2 (\simeq \mathbb{P}^2)$. Looking at the incidence correspondence we have the following diagram:

$$\begin{tikzcd} \overline{M}_{0,1}(V_5,1) \arrow[r, "\mathrm{ev}_1", "3:1"'] \arrow[d, "\varphi"'] & V_5 \\ F_1(V_5) \simeq \mathbb{P}^2 \end{tikzcd} \quad . \tag{1.B}$$

The map ev_1 is a 3-to-1 cover ramified on the surface E and fully ramified on the sextic σ (cf. [Ili94, 1.2.1(3)], [San14, Corollary 2.24]).

Let $F_2(V_5)$ be the Fano varieties of lines and conics on V_5. Then $F_2(V_5)$ is isomorphic to $\mathbb{P}^4$ (cf. [Ili94, Proposition 1.22]). Let $M = \overline{M}^{\mathrm{bir}}_{0,1}(V_5,2)$ be the Kontsevich moduli space of genus 0 degree 2 stable maps with one marked point and birational onto their image. We have the following diagram

$$\begin{tikzcd} M \arrow[r, "\mathrm{ev}"] \arrow[d, "\pi"'] & V_5 \\ F_2(V_5). \end{tikzcd} \tag{1.C}$$

We will consider the moduli spaces $\overline{M}_{0,m}(V_5,d)$ for $d \geq 2$ and prove some rational connectedness, unirationality and rationality results. Quite recently, the structure of $\overline{M}_{0,0}(V_5,d)$ has been studied in [LT17, Theorem 7.9]. In particular, its decomposition in irreducible components is deduced. Our approach is different for V_5 and we will give another proof of the irreducibility of $\overline{M}^{\mathrm{bir}}_{0,0}(V_5,d)$ that also proves that this variety is unirational of dimension $2d$. From this the structure of the irreducible components of $\overline{M}_{0,0}(V_5,d)$ also follows. This will be discussed in Section 3. For the moment we state the results of [LT17].

Definition 1.4. For any $d \geq 1$ the closure in $\overline{M}_{0,0}(V_5, d)$ of the moduli space of morphisms which factor through a line is denoted by $\overline{M}_{0,0}^{\text{line}}(V_5, d)$.

Theorem 1.5 ([Isk79][LT17])**.** *For any $d \geq 1$ the moduli space $\overline{M}_{0,0}(V_5, d)$ has pure dimension $2d$.*

(1) *For $d = 1$, the moduli space $\overline{M}_{0,0}(V_5, d)$ is isomorphic to $\mathbb{P}^2$ (see [Isk79]).*
(2) *For $d \geq 2$, the moduli space $\overline{M}_{0,0}(V_5, d)$ has two irreducible components (see [LT17]): $\overline{M}_{0,0}(V_5, d) = \overline{M}_{0,0}^{\text{bir}}(V_5, d) \cup \overline{M}_{0,0}^{\text{line}}(V_5, d)$.*

We will later on prove the unirationality of the first component and the rationality of the seconde one, see Theorem 3.1.

2. Strong rational simple connectedness

In this section, we prove the following theorem.

Theorem 2.1. *Let $m \geq 0$ and let $d \geq 1$.*

(1) *The moduli space $\overline{M}_m^{\text{bir}}(V_5, d)$ is rationally connected.*
(2) *If $m = 1$ and $d \geq 6$, the general fibers of the map $\mathrm{ev} : \overline{M}_m^{\text{bir}}(V_5, d) \to V_5^m$ are rationally connected.*
(3) *If $m \geq 2$ and $d \geq 8m - 6$ for d even and $d \geq 8m - 1$ for d odd, the general fibers of the map $\mathrm{ev} : \overline{M}_m^{\text{bir}}(V_5, d) \to V_5^m$ are rationally connected.*

Remark 2.2. The above bounds for $m \geq 1$ are not optimal.

(1) For $m = 1$ or $m = 2$, we will prove in the next Section, see Theorem 3.1, that the fiber is even unirational for all $d \geq 2$.
(2) The dimension of degree d curves passing through m points is given by $2d + 3 + m - 3 - 3m = 2d - 2m = 2(d - m)$. So a possible bound would be $d = m$ but Gromov-Witten computations show that this is not possible: the fiber is finite dimensional but not irreducible for large d.

2.1. Lifting curves through ev.

Lemma 2.3. *Let $f : \mathbb{P}^1 \to V_5$ be a general element in $\overline{M}_0^{\text{bir}}(V_5, d)$ and let $W = \mathbb{C}^5/V_2$ where V_2 is the tautological subbundle of $\mathrm{Gr}(2, 5)$. Then we have*

$$f^*W = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c)$$

with $a + b + c = d$, $a \geq b \geq c$ and $a - c \leq 1$.

Proof. The decomposition holds for any map without the last condition. We only need to prove that $a - c \leq 1$ for a general element. Note that for $d \in \{1, 2, 3\}$, the statement holds since V_5 contains irreducible conics and twisted cubics. Note that the statement is equivalent to $H^1(\mathbb{P}^1, f^*\mathrm{End}(W)) = 0$. In particular we only need to find a stable map $f : C \to V_5$ in $\overline{M}_0^{\text{bir}}(V_5, d)$ with $H^1(C, f^*\mathrm{End}(W)) = 0$.

We proceed by induction on $d \geq 4$. Consider a stable map $f : C \to V_5$ with $C = C_1 \cup C_2$ where C_1 and C_2 meet in one point p_0, such that $f|_{C_1}$ is the inclusion of a general twisted cubic in V_5, $\deg(f|_{C_2}) = d-3$ and $H^1(C_2, (f|_{C_2})^*\mathrm{End}(W)) = 0$. We use the exact sequence

$$0 \to f^*\mathrm{End}(W)|_{C_1}(-p_0) \to f^*\mathrm{End}(W) \to f^*\mathrm{End}(W)|_{C_2} \to 0.$$

Since C_1 is a twisted cubic, we have $f^*W|_{C_1} \simeq \mathcal{O}_{\mathbb{P}^1}(1)^3$ and $f^*\mathrm{End}(W)|_{C_1}(-p_0) \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^9$. We get the vanishing of $H^1(C, f^*\mathrm{End}(W))$. $\square$

Proposition 2.4. *Let $d \geq 1$ and $D \geq \lfloor \frac{d}{3} \rfloor$.*

Let $f : \mathbb{P}^1 \to V_5$ be a general element in $\overline{M}_0^{\mathrm{bir}}(V_5, d)$. Then there exists a map $\widetilde{f} : \mathbb{P}^1 \to M$ of bidegree (D, d) with $f = \widetilde{f} \circ \mathrm{ev}$. Furthermore, the set of such liftings is birational to a projective space of dimension $3D - d + 2$.

Proof. The map ev is the pull-back of the map $\mathrm{Fl}(2,4;5) \to \mathrm{Gr}(2,5)$ along the closed embedding $V_5 \subset \mathrm{Gr}(2,5)$. In particular ev is a $\mathbb{P}^2$-bundle with $\mathbb{C}^5/V_2$ as associated rank 3 vector bundle where V_2 is the tautological subbundle on $\mathrm{Gr}(2,5)$. Finding a lifting $\widetilde{f} : \mathbb{P}^1 \to M$ of degree (D,d) of the degree d map $f : \mathbb{P}^1 \to V_5$ is equivalent to finding a surjective map $f^*(\mathbb{C}^5/V_2) \to \mathcal{O}_{\mathbb{P}^1}(D)$. We have $f^*(\mathbb{C}^5/V_2) = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c)$ with $a + b + c = d$ and $a \geq b \geq c$. Such a surjective map exists as soon as $D = c$ or $D \geq b$. Furthermore, for a general element f, we have $a - c \leq 1$ (see Lemma 2.3) proving that $D \geq a$ or $D = c$. The set of such lifting is an open subset of $\mathbb{P}(\mathrm{Hom}(f^*(\mathbb{C}^5/V_2), \mathcal{O}_{\mathbb{P}^1}(D))$ which is a projective space of dimension $3D - d + 2$. ∎

Corollary 2.5. *Let $d \geq 1$ and $D \geq \lfloor \frac{d}{3} \rfloor$.*

(1) *The moduli space $\overline{M}_{0,m}(M, (D,d))$ contains a unique irreducible component, denoted by $\overline{M}_{0,m}^{\mathrm{bir}}(M, (D,d))$, dominating $\overline{M}_m^{\mathrm{bir}}(V_5, d)$. This component is of dimension $3D + d + m + 2$.*

(2) *Let $\widetilde{f} \in \overline{M}_{0,m}(M, (D,d))$ with irreducible source and such that $\mathrm{ev} \circ \widetilde{f}$ lies in $\overline{M}_m^{\mathrm{bir}}(V_5, d)$ and satisfies the assumption in Lemma 2.3. Then $\widetilde{f}$ lies in $\overline{M}_{0,m}^{\mathrm{bir}}(M, (D,d))$.*

Proof. (1) The map $\overline{M}_{0,m}(M, (D,d)) \to \overline{M}_{0,m}(V_5, d)$ is obtained from the morphism $\overline{M}_{0,m}(\mathrm{Fl}(2,4;5), (D,d)) \to \overline{M}_{0,m}(\mathrm{Gr}(2,5), d)$ by base change. Furthermore, the last two moduli spaces are irreducible, smooth and of expected dimensions $3D + 4d + M + 5$ and $5d + m + 3$.

Let $\phi : \overline{M}_{0,m+1}(\mathrm{Gr}(2,5), d) \to \overline{M}_{0,m}(\mathrm{Gr}(2,5), d)$ be the universal curve and $\mathrm{ev} : \overline{M}_{0,m+1}(\mathrm{Gr}(2,5), d) \to V_5$ be the evaluation map. Let $\mathring{M}$ be the open subset of $\overline{M}_{0,m}(\mathrm{Gr}(2,5), d)$ of maps with irreducible source such that $f^*(\mathbb{C}^5/V_2) = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c)$ with $a + b + c = d$, $a \geq b \geq c$ and $a - c \leq 1$. Set $E = \phi_*(\mathrm{ev}^*(\mathbb{C}^5/V_2)^\vee \otimes \mathcal{O}_\phi(1)^{\otimes D})$ where $\mathcal{O}_\phi(1)$ is the relative ample generator over $\mathring{M}$. Then E is a vector bundle and the moduli space $\overline{M}_{0,m}(\mathrm{Fl}(2,4;5), (D,d))$ is birational to an open subset of $\mathbb{P}_{\mathring{M}}(E)$. In particular the map $\overline{M}_{0,m}^{\mathrm{bir}}(M, (D,d)) \to \overline{M}_m^{\mathrm{bir}}(V_5, d)$ is an open subset of the projective bundle $\mathbb{P}_{\overline{M}_m^{\mathrm{bir}}(V_5,d) \cap \mathring{M}}(E)$. Since $\overline{M}_m^{\mathrm{bir}}(V_5, d)$ is irreducible, so is $\overline{M}_{0,m}^{\mathrm{bir}}(M, (D,d))$. The dimension of $\overline{M}_m^{\mathrm{bir}}(V_5, d)$ is $2d + m$ while the fiber is of dimension $3D - d + 2$ giving the dimension formula.

(2) Let $\widetilde{f} \in \overline{M}_{0,m}(M, (D,d))$ with irreducible source and such that $\mathrm{ev} \circ \widetilde{f}$ satisfies the assumption in Lemma 2.3. Then $\mathrm{ev} \circ \widetilde{f} \in \overline{M}_m^{\mathrm{bir}}(V_5, d) \cap M$ and $\widetilde{f}$ is a point of the above projective bundle and therefore lies in $\overline{M}_{0,m}^{\mathrm{bir}}(M, (D,d))$. ∎

Remark 2.6. We make two remarks concerning last proof:

(1) The moduli space $\overline{M}_{0,m}^{\mathrm{bir}}(M, (D,d))$ is actually unirational as a projective bundle over $\overline{M}_m^{\mathrm{bir}}(V_5, d)$ which we prove to be unirational in Theorem 3.1

(2) For a projective bundle $p : Y \to Z$ the fact that the map between moduli spaces of stable maps is an open subset of a projective bundle is classical, see for example [Per02, Proposition 4] or [GPPS18, Lemma 2.1].

2.2. **Lifting curves through π.**

Proposition 2.7. *Let $d \geq 1$ and $D \geq \lfloor \frac{d}{3} \rfloor$.*

Let $\overline{f} : \mathbb{P}^1 \to F_2(V_5)$ be a general element in $\overline{M}_{0,0}(F_2(V_5), D)$. Then there exists an $\widetilde{f} \in \overline{M}_{0,0}(M, (D, d))$ of bidegree (D, d) with $\overline{f} = \widetilde{f} \circ \pi$ if and only if $d \geq 2D - 1$. Furthermore $\widetilde{f}$ can be chosen in $\overline{M}^{\rm bir}_{0,0}(M, (D, d))$

Proof. Since $F_2(V_5)$ is isomorphic to $\mathbb{P}^4$ and since $\overline{M}_{0,0}(\mathbb{P}^4, D)$ is irreducible (see [Tho98, KP01, Per02]), the moduli space $\overline{M}_{0,0}(F_2(V_5), D)$ is irreducible and to prove existence, it is enough to prove that the differential map, which is given by

$$H^0(\mathbb{P}^1, \widetilde{f}^* T_M) \to H^0(\mathbb{P}^1, \overline{f}^* T_{F_2(V_5)}),$$

is surjective. To prove last assertion we compute this map for $\widetilde{f} \in \overline{M}^{\rm bir}_{0,0}(M, (D, d))$.

We have an exact sequence $0 \to T_\pi \to T_M \to \pi^* T_{F_2(V_5)} \to 0$ and it therefore suffices to prove the vanishing of the cohomology group $H^1(\mathbb{P}^1, \overline{f}^* T_\pi)$. We therefore compute T_π.

Since V_5 is a codimension 3 linear section of $\mathrm{Gr}(2, 5)$, we have an exact sequence $0 \to T_{V_5} \to T_{\mathrm{Gr}(2,5)} \to \mathcal{O}_{V_5}(1)^3 \to 0$. By pull-back, we get an exact sequence $0 \to T_M \to T_{\mathrm{Fl}(2,4;5)} \to \mathrm{ev}^* \mathcal{O}_{V_5}(1)^3 \to 0$. Denote by p and q the morphisms $\mathrm{Fl}(2, 4; 5) \to \mathrm{Gr}(2, 5)$ and $\mathrm{Fl}(2, 4; 5) \to \mathbb{P}^4$. The map q induces an exact sequence $0 \to T_q \to T_{\mathrm{Fl}(2,4;5)} \to q^* T_{\mathbb{P}^4} \to 0$ and we get

$$0 \to T_\pi \to T_q \to \mathrm{ev}^* \mathcal{O}_{V_5}(1)^3 \to 0.$$

Note that T_π is a line bundle so we only need to compute its first Chern class. Now if V_2 and V_4 are the tautological subbundles of $\mathrm{Gr}(2, 5)$ and $\mathrm{Gr}(4, 5) = \mathbb{P}^4$, we have $T_{\mathrm{Fl}(2,4;5)} = \mathrm{Ker}(V_2^\vee \otimes (\mathbb{C}^5/V_2) \oplus (V_4^\vee \otimes (\mathbb{C}^5/V_4) \to V_2^\vee \otimes (\mathbb{C}^5/V_4))$ and $T_{\mathrm{Gr}(4,5)} = V_4^\vee \otimes (\mathbb{C}^5/V_4)$ so that $\det(T_q) = q^*\mathcal{O}_{\mathbb{P}^4}(-2) \otimes p^*\mathcal{O}_{\mathrm{Gr}(2,5)}(4)$ so that $T_\pi = \pi^*\mathcal{O}_{\mathbb{P}^4}(-2) \otimes \mathrm{ev}^* \mathcal{O}_{V_5}(1)$. We get $\overline{f}^* T_\pi = \mathcal{O}_{\mathbb{P}^1}(d - 2D)$ and the desired vanishing for $d \geq 2D - 1$.

To prove the converse, first note that we may assume that $\overline{f}$ has irreducible source. If $\widetilde{f}$ has a non irreducible source, then taking the component that surject onto the source of $\overline{f}$ we may assume that $\widetilde{f}$ also has an irreducible source. Now since a general element $\overline{f}$ has a lift, the map $\overline{M}_{0,0}(M, (D, d)) \to \overline{M}_{0,0}(\mathbb{P}^4, D)$ must be surjective and therefore the differential map must be surjective on an open subset. By the previous computation this is possible only for $d \geq 2D - 1$. $\square$

For $\overline{f} : \mathbb{P}^1 \to F_2(V_5)$ a degree D morphism, let $\pi_{\overline{f}} : S_{\overline{f}} = \mathbb{P}^1 \times_{F_2(V_5)} M \to \mathbb{P}^1$ be the corresponding conic bundle.

Proposition 2.8. *Let d and D such that $d \geq 2D - 1$ and $D \geq \lfloor \frac{d}{3} \rfloor$.*

Let $\overline{f} : \mathbb{P}^1 \to F_2(V_5)$ be a general degree D morphism and let $\widetilde{f} \in \overline{M}^{\rm bir}_{0,0}(M, (D, d))$ of bidegree (D, d) with $\overline{f} = \widetilde{f} \circ \pi$.

(1) *The surface $S_{\overline{f}}$ is a smooth rational surface its image in V_5 has degree $5D$.*
(2) *The conic bundle $\pi_{\overline{f}} : S_{\overline{f}} \to \mathbb{P}^1$ has $3D$ singular fibers $(\mathcal{F}_i)_{i \in [1,3D]}$. Each $\mathcal{F}_i$ is the union of two distinct lines: $\mathcal{F}_i = L_i \cup L'_i$.*

(3) *The curve* $\Gamma = \overline{f}(\mathbb{P}^1)$ *meets* $\mathcal{F}_i$ *in a smooth point. Denote by* $(L_i)_{i\in[1,3D]}$ *the lines meeting* Γ *and by* $(L'_i)_{i\in[1,3D]}$ *the other lines.*
(4) *On* $S_{\overline{f}}$ *we have* $\Gamma^2 = d - 2D$.
(5) *The map* $\mathrm{ev} : T_{S_{\overline{f}}} \to \mathrm{ev}^* T_{V_5}$ *is an injective map of vector bundles outside of a finite set of points.*

Proof. (1) The surface $S_{\overline{f}}$ is clearly rational. Furthermore, since M is smooth, by Kleimann-Bertini Theorem [Kle74], the fiber product $S_{\overline{f}}$ is smooth for $\overline{f}$ general.

Let $L \subset F_2(V_5)$ be a general line. The variety $p(q^{-1}(L)) \subset \mathrm{Gr}(2,5)$ is a Schubert variety of codimension 1 in $\mathrm{Gr}(2,5)$. It is therefore an hyperplane section. Therefore $\mathrm{ev}(\pi^{-1}(L))$ is an hyperplane section of V_5 and therefore of degree 5. This implies that $\mathrm{ev}(S_{\overline{f}})$ has degree $5D$.

(2) We need to prove that the conic bundle $\pi : M \to F_2(V_5)$ has singular fibers over a hypersurface of degree 3 in $F_2(V_5)$ and non reduced fibers in codimension at least two. To prove this we consider a line $L \subset F_2(V_5)$ of conics in V_5. The variety $p(q^{-1}(L)) \subset \mathrm{Gr}(2,5)$ is a Schubert variety of codimension 1 in $\mathrm{Gr}(2,5)$. It is therefore an hyperplane section. Therefore $\mathrm{ev}(\pi^{-1}(L))$ is an hyperplane section of V_5. A general such line defines a general hyperplane section *i.e.* a Del Pezzo surface of degree 5. On such a Del Pezzo surface Σ, there are five families of conics. In fact considering Σ as the blow-up of $\mathbb{P}^2$ in four points $(x_i)_{i\in[1,4]}$ the families are given by

- the conics through the four points x_1, x_2, x_3, x_4 ;
- the lines through one of the four points (4 families).

The family defined by the image via ev of the fibers of $\pi : \pi^{-1}(L) \to L$ is one of these five families. But in each of these families, there is no non-reduced conic and exactly 3 degenerate conics that are union of two distinct lines: in the first family we have the three union of lines $(x_ix_j) \cup (x_kx_l)$ for $\{i,j\} \coprod \{k,l\} = \{1,2,3,4\}$ a partition. In the family of lines passing through x_1, we have the union of the exceptional divisor over x_1 and one of the line (x_1x_i) for $i \in \{2,3,4\}$.

(3) This follows from the fact that Γ is a section of $\pi_{\overline{f}}$.

(4) Blowing-down the surface $S_{\overline{f}}$ along the lines $(L_i)_{i\in[1,3D]}$ defines a rational ruled surface. In particular, the Picard group $\mathrm{Pic}(S_{\overline{f}})$ of $S_{\overline{f}}$ is generated by the class of Γ, the class F of a general fiber of $\pi_{\overline{f}}$ and the classes of the lines $(L_i)_{i\in[1,3D]}$.

Let H be class of an effective divisor associated to $\mathcal{O}_{V_5}(1)$ on $S_{\overline{f}}$ and write H in the previous basis. We get

$$H = a\Gamma + bF + \sum_{i=1}^{3D} a_i L_i.$$

We compute these coefficients. Since ev maps F to a conic in V_5, we get $a = H \cdot F = 2$. Since ev maps L_i to a line in V_5, we get $1 = H \cdot L_i = a - a_i$ thus $a_i = 1$. Since ev maps Γ to a curve of degree d in V_5, we get $H \cdot \Gamma = d$. Since ev maps $S_{\overline{f}}$ to a surface of degree $5D$, we have $H^2 = 5D$. Solving the system obtained we get $b = D - d$ and $\Gamma^2 = d - 2D$.

(5) The map $T_{\mathrm{Fl}(2,4;5)} \to \mathrm{ev}^* T_{\mathrm{Gr}(2,5)}$ is always injective along the fibers of the map $p : \mathrm{Fl}(2,4;5) \to \mathrm{Gr}(4,5) = \mathbb{P}^4$ so that its kernel is mapped isomorphically into $T_{\mathrm{Gr}(4,5)}$. Furthermore, at a point $(V_2, V_4) \in \mathrm{Fl}(2,4;5)$ corresponding to subspaces of dimension 2 and 4, the kernel of the map $T_{\mathrm{Fl}(2,4;5)} \to \mathrm{ev}^* T_{\mathrm{Gr}(2,5)}$ seen as a subspace

of $T_{\mathrm{Gr}(4,5)}$ is given by the tangent space of the plane $\mathbb{P}^2 = \{W_4 \in \mathrm{Gr}(4,5) \mid W_4 \supset V_2\}$.

For a given $V_4 \in \mathrm{Gr}(4,5) = F_2(V_5)$, we define an hypersurface $Q_{V_4} \subset F_2(V_5)$ via $Q_{V_4} = \pi(\mathrm{ev}^{-1}(\mathrm{ev}(\pi^{-1}(V_4))))$. Denote by C_{V_4} the conic associated to V_4 *i.e* $C_{V_4} = \mathrm{ev}(\pi^{-1}(V_4))$. We have

$$Q_{V_4} = q(p^{-1}(C_{V_4}) = \{W_4 \in F_2(V_5) \mid \text{ there exists } V_2 \in C_{V_4} \text{ such that } W_4 \supset V_2\}.$$

Consider the surface $S_{\overline{f}}$ and the map $T_{S_{\overline{f}}} \to \mathrm{ev}^* T_{V_5}$. By the previous discution, this map is always injective along the fibers of π and is not injective at (V_2, V_4) only if the map $T_{\mathbb{P}^1} = \mathrm{d}\pi(T_{S_{\overline{f}}}) \to T_{F_2(V_5)}$ is not injective at V_4. This is the case if and only if the tangent of $\overline{f}(\mathbb{P}^1)$ at V_4 is contained in Q_{V_4}. Since $\overline{f}$ is chosen general, this occurs on a closed subset of $\mathbb{P}^1$. We may also assume that this occurs outside of the locus where π has singular fibers. Finally, at a point $V_4 \in \overline{f}(\mathbb{P}^1)$ where this occurs, the tangent line is contained in exactly two planes of the form $\mathbb{P}^2 = \{W_4 \in \mathrm{Gr}(4,5) \mid W_4 \supset V_2\}$. So, over a point $V_4 \in \overline{f}(\mathbb{P}^1)$ where this occurs, there are exactly two points (V_2, V_4) and (V_2', V_4) of the surface S_{of} where $T_{S_{\overline{f}}} \to \mathrm{ev}^* T_{V_5}$ is not injective. This prove that the map $T_{S_{\overline{f}}} \to \mathrm{ev}^* T_{V_5}$ is injective outside of a finite number of points. □

Proposition 2.9. *Let d and D such that $d \geq 2D - 1$ and $D \geq \lfloor \frac{d}{3} \rfloor$.*

Let $\overline{f} : \mathbb{P}^1 \to F_2(V_5)$ be a general degree D morphism and let $\widetilde{f} \in \overline{M}_{0,0}^{\mathrm{bir}}(M, (D,d))$ be general of bidegree (D,d) with $\overline{f} = \widetilde{f} \circ \pi$. Let $\Gamma = \widetilde{f}(\mathbb{P}^1)$ and choose m general fibers $F_1, \cdots, F_m$ of the map $\pi_{\overline{f}} : S_{\overline{f}} \to \overline{f}(\mathbb{P}^1)$. Let $(p_i)_{i \in [1,m]}$ be the intersection points of Γ with F_i: $p_i = \Gamma \cap F_i$. Choose m points $(q_i)_{i\in[1,m]}$ such that $q_i \in F_i \setminus \{p_i\}$.

Set $C' = \Gamma \cup \bigcup_{i=1}^m F_i$ and let $f' = (\mathrm{ev} : C' \to V_5, (q_i)_{i\in[1,m]}) \in \overline{M}_m^{\mathrm{bir}}(V_5, d)$.

Then f' is a smooth point in its fiber of the map $\mathrm{ev} : \overline{M}_m^{\mathrm{bir}}(V_5, d) \to V_5^m$.

Proof. Let $\Sigma = \sum_i q_i$. It suffices to prove that $H^1(C', \mathrm{ev}^* T_{V_5}(-\Sigma))$ vanishes. Since we choose general elements, we may assume that $\mathrm{ev}(\Gamma)$ meets the open $\mathrm{SL}_2(\mathbb{C})$-orbit and that the conics F_i are general.

First remark that for a general conic $F \subset \mathrm{Gr}(2,5)$, we have $T_{\mathrm{Gr}(2,5)}|_{V_5} = \mathcal{O}_F(1)^2 \oplus \mathcal{O}_F(2)$ (since there is a unique conic passing through two points in general position in V_5).

Consider the exact sequence

$$0 \to \bigoplus_{i=1}^m T_{V_5}|_{F_i}(-\Sigma - p_i) \to T_{V_5}|_{C'}(-\Sigma) \to T_{V_5}|_\Gamma(-\Sigma) \to 0.$$

Since Σ is not meeting Γ, we have $T_{V_5}|_\Gamma(-\Sigma) = T_{V_5}|_\Gamma$ and since T_{V_5} is globaly generated on an open subset (the open $\mathrm{SL}_2(\mathbb{C})$-orbit) we have $H^1(\Gamma, T_{V_5}|_\Gamma(-\Sigma)) = H^1(\Gamma, T_{V_5}|_\Gamma) = 0$. On the other hand, the intersection $\Sigma \cap F_i$ contains a unique point: q_i so that $T_{V_5}|_{F_i}(-\Sigma - p_i) = T_{V_5}|_{F_i}(-q_i - p_i) = \mathcal{O}_{F_i}(-1) \oplus \mathcal{O}_{F_i}^2$ and we get $H^1(F_i, T_{V_5}|_{F_i}(-\Sigma - p_i)) = 0$ proving the vanishing. □

2.3. **Family of pairs.** We study compatible pairs of sections of π. Fix d and m and define the following integers: $d_0 = d - 2m$, $D = \lfloor \frac{d_0}{2} \rfloor$, $d_E = 4D - d_0$ and $h = 2D - d_E = d_0 - 2D$. Finally set $m' = d - 2D - m = m + h$.

Hypothesis 2.10. *We assume that the following conditions are satisfied:*

(1) $m' \geq 1$,

(2) $d_0 \geq h+2$,
(3) $d \geq 8m - 6 + 5h$.

Remark 2.11. The above definition depends on the parity of d.

(1) We summarise the different values of d_0, D, d_E, h and m'.

	$d=2k$	$d=2k+1$
d_0	$2(k-m)$	$2(k-m)+1$
D	$k-m$	$k-m$
d_E	$2(k-m)$	$2(k-m)-1$
h	0	1
m'	m	$m+1$

(2) We have $d - d_E = 2d - 4D - 2m = 2m'$.
(3) We have $D - \lceil \frac{d_E}{3} \rceil \geq m' - 1$
(4) We have $D > \frac{d}{3} - 1$ and therefore $D \geq \lfloor \frac{d}{3} \rfloor$.
(5) We have $d_E \geq 2D - 1$.

Remark 2.12. Let d and m be non negative integers.

(1) If $m = 0$, then (d,m) satisfy Hypothesis 2.10 for $d \geq 3$.
(2) If $m = 1$, then (d,m) satisfy Hypothesis 2.10 as soon as $d \geq 6$.
(3) If $m \geq 2$, then (d,m) satisfy Hypothesis 2.10 for $d \geq \begin{cases} 8m-6 \text{ for } d \text{ even,} \\ 8m-1 \text{ for } d \text{ odd.} \end{cases}$

We have seen in Propositions 2.7 and 2.8 that for $\overline{f} \in \overline{M}_{0,m}(F_2(V_5), D)$ general (in particular with irreducible source), there exists $\widetilde{f}_E \in \overline{M}^{\mathrm{bir}}_{0,m}(M,(D,d_E))$ with irreducible source such that $\overline{f} = \widetilde{f}_E \circ \pi$. We obtain this way a conic bundle surface $S_{\overline{f}} \to \mathbb{P}^1$ with a section $E = \widetilde{f}_E(\mathbb{P}^1)$. This conic bundle has $3D$ singular fibers, the section E meets exactly one of these lines in each singular fiber and we have $E^2 = d_E - 2D = -h$.

Lemma 2.13. *For E as above, let $\overline{S}_{\overline{f},E}$ be the ruled surface obtained from $S_{\overline{f}}$ by contraction of the $3D$ lines in the singular fibers of $S_{\overline{f}} \to \mathbb{P}^1$ not meeting E. Then $\overline{S}_{\overline{f},E}$ is a ruled surface and the image $\overline{E}$ of E in $\overline{S}_{\overline{f},E}$ is a section with minimal self intersection. In particular $\overline{S}_{\overline{f},E}$ is a Hirzebruch surface of type h.*

Proof. This is classical geometry of surfaces. □

In the next proposition, we will use the following lemma which is a direct application of general techniques (see [Kol96, Theorem II.1.7]).

Lemma 2.14. *Let $\mathcal{F}$ be a vector bundle of rank r on $\mathbb{P}^1$ such that $\mathcal{F} = \oplus_{i=1}^m \mathcal{O}_{\mathbb{P}^1}(a_i)$, with $a_1 \geq \cdots \geq a_r \geq 0$. Let $m \geq 0$, let $(x_i)_{i\in[1,m]}$ be points on $\mathbb{P}^1$ and let $y_i \in \mathcal{F}_{x_i}$ be elements in the stalk of $\mathcal{F}$ at x_i for all $i \in [1,m]$.*

Assume such that $a_r \geq m-1$, then the space

$$V = \{s \in H^0(\mathbb{P}^1,\mathcal{F}) \mid s(x_i) = y_i \text{ for all } i \in [1,m]\}$$

has dimension $\dim H^0(\mathbb{P}^1,\mathcal{F}) - rm$.

Proof. Let $\pi : Y \to \mathbb{P}^1$ be the total space of $\mathcal{F}$. The space V is the scheme of morphisms $s : \mathbb{P}^1 \to Y$ relative over $\mathbb{P}^1$ with condition $s(x_i) = y_i$ for all $i \in [1,m]$. Its obstuction space is $H^1(\mathbb{P}^1, s^*T_\pi(-m)) = H^1(\mathbb{P}^1,\mathcal{F}(-m))$ and vanishes under our assumption. It is therefore smooth (it is a vector space) of dimension $\dim H^0(\mathbb{P}^1, s^*T_\pi(-m)) = \dim H^0(\mathbb{P}^1, \mathcal{F}(-m))$. Under our assumption, this last space has dimension $\dim H^0(\mathbb{P}^1, \mathcal{F}) - mr$. $\square$

Set $\overline{M}(d,m) = \overline{M}^{bir}_{0,m'}(M,(D,d_E)) \times_{\overline{M}_{0,m'}(F_2(V_5),D)} \overline{M}_{0,m'}(M,(D,d))$. Define $M(d,m)$ as the open subset of $\overline{M}(d,m)$ of pairs $(\widetilde{f}_E, \widetilde{f})$ such that the stable map $\widetilde{f}_E \in \overline{M}^{\mathrm{bir}}_{0,m'}(M,(D,d_E))$ has irreducible source and satisfies all properties of Proposition 2.9. We consider a closed subset $\mathcal{M}_{d,m} \subset \overline{M}(d,m)$ defined as the closure of the set

$$\mathcal{M}_{d,m} = \overline{\left\{(\widetilde{f}_E, \widetilde{f}) \in M(d,m) \;\middle|\; [\widetilde{f}(\mathbb{P}^1)] = [\widetilde{f}_E(\mathbb{P}^1)] + m'[F] \text{ in } \operatorname{Pic}(S_{\overline{f}})\right\}}$$

where $\overline{f} = \widetilde{f} \circ \pi = \widetilde{f}_E \circ \pi$ and F is a general fiber of the map $\pi : S_{\overline{f}} \to \mathbb{P}^1$.

Proposition 2.15. *Let m, d, d_0, D, d_E and m' as above.*

(1) *The space $\mathcal{M}_{d,m}$ is irreducible of dimension $d + d_E + D + m' + 3$.*
(2) *For $(\widetilde{f}_E, \widetilde{f}) \in \mathcal{M}_{d,m}$, we have $\widetilde{f} \in \overline{M}^{\mathrm{bir}}_{0,m'}(M,(D,d))$.*
(3) *The map $\mathcal{M}_{d,m} \to \overline{M}^{\mathrm{bir}}_{0,m'}(M,(D,d))$ is surjective.*

Proof. (1) We have a morphism $\mathcal{M}_{d,m} \to \overline{M}^{\mathrm{bir}}_{0,m'}(M,(D,d_E))$ whose general fibers are given by the linear system $|\widetilde{f}_E(\mathbb{P}^1) + m'F|$. On the open subset $M(d,m)$, set $E = \widetilde{f}_E(\mathbb{P}^1) \subset S_{\overline{f}}$ and consider $\overline{S}_{\overline{f},E}$ the ruled surface obtained by contracting the lines in the singular fibers of $S_{\overline{f}} \to \mathbb{P}^1$ not meeting E. Then the above linear system is equal to $|\overline{E} + m'\overline{F}|$. where $\overline{E}$ and $\overline{F}$ are the images of E and F in $\overline{S}_{\overline{f},E}$. Pushing down along the map $\overline{S}_{\overline{f},E} \to \mathbb{P}^1$ we get that this linear system is the projective space on the cohomology group $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(m'))$. This realises $\mathcal{M}(d,m)$ as an open subset of a projective bundle over $\overline{M}^{\mathrm{bir}}_{0,m'}(M,(D,d_E))$ proving the irreducibility (see below for the construction of this bundle). The dimension is given by

$$\begin{aligned}\dim \mathcal{M}(d,m) &= \dim \overline{M}^{\mathrm{bir}}_{0,m'}(M,(D,d_E)) + m + m' + 1\\ &= 3D + d_E + m' + 2 + d - 2D + 1\\ &= d + D + d_E + m' + 3.\end{aligned}$$

We describe the above projective bundle on an open subset of the moduli space $\overline{M}^{\mathrm{bir}}_{0,m'}(M,(D,d_E))$. Let $\overline{\mathcal{M}}$ be the image in $\overline{M}_{0,m'}(F_2(V_5),D)$ of $M(d,m)$ and let $\mathcal{M}$ be its inverse image in $\overline{M}^{\mathrm{bir}}_{0,m'}(M,(D,d_E))$. Let $p : \mathcal{C} \to \mathcal{M}$ be the universal curve. Define $\mathcal{S} = \mathcal{C} \times_{F_2(V_5)} M$ where the map $\mathcal{C} \to F_2(V_5)$ is the composition of the evaluation $\mathcal{C} \to M$ with $\pi : M \to F_2(V_5)$. By construction, we have a section $s : \mathcal{C} \to \mathcal{S}$ of $\mathrm{pr} : \mathcal{S} \to \mathcal{C}$. Furthermore, by definition we have sections $\sigma_i : \mathcal{M} \to \mathcal{C}$ for all $i \in [1, m']$. Define the divisor $\mathcal{D}$ on $\mathcal{S}$ by

$$\mathcal{D} = s(\mathcal{C}) + \sum_{i=1}^{m'} \mathrm{pr}^{-1}(\sigma_i(\mathcal{M}))$$

and set $\mathcal{E} = p_*\mathrm{pr}_*\mathcal{O}_{\mathcal{S}}(\mathcal{D})$. By the above, this is a vector bundle over $\mathcal{M}$ and the projective bundle is $\mathbb{P}_{\mathcal{M}}(\mathcal{E})$.

(2) Note that for a general element $\widetilde{f}_E \in \overline{M}^{\rm bir}_{0,m'}(M,(D,d_E))$, the line bundle $\mathcal{O}_{\overline{S}_{\overline{f},E}}(\overline{E}+m'\overline{F})$ is globally generated. In particular a general element $\widetilde{f}$ such that $(\widetilde{f}_E,\widetilde{f}) \in \mathcal{M}(d,m)$ has for image in $\overline{S}_{\overline{f},E}$ a general section of $H^0(\overline{S}_{\overline{f},E},\mathcal{O}_{\overline{S}_{\overline{f},E}}(\overline{E}+m'\overline{F}))$ which is irreducible so that $\widetilde{f}$ has irreducible source. Furthermore, we have seen that on $S_{\overline{f}}$, a section H of $\mathrm{ev}^*\mathcal{O}_{V_5}(1)$ has class

$$H = 2\widetilde{f}_E(\mathbb{P}^1) + (D-d_E)F + \sum_{i=1}^{3D} L_i$$

so that its image in $\overline{S}_{\overline{f},E}$ is $\overline{H} = 2\overline{E}+(D-d_E+3D)\overline{F} = 2\overline{E}+d_0\overline{F}$. Since $d_0 \geq 2+h$, the line bundle $\overline{H}$ is very ample and the map $\mathrm{ev} : S_{\overline{f}} \to V_5$ is an isomorphism onto its image except maybe on the $3D$ lines contrated by the map $S_{\overline{f}} \to \overline{S}_{\overline{f},E}$. In particular, the map $f = \mathrm{ev}\circ\widetilde{f}$ is birational onto its image so lies in $\overline{M}^{\rm bir}_{0,m'}(V_5,d)$.

Now let M_0 be the image of $\mathcal{M}(d,m)$ in $\overline{M}^{\rm bir}_{0,m'}(V_5,d)$ via the map $(\widetilde{f}_E,\widetilde{f}) \mapsto \mathrm{ev}\circ\widetilde{f}$. We want to prove that M_0 is $\overline{M}^{\rm bir}_{0,m'}(V_5,d)$. Consider the maps

$$\mathcal{M}_{d,m} \stackrel{\alpha}{\to} \overline{M}_{0,m'}(M,(D,d)) \stackrel{\beta}{\to} M_0$$

defined by $(\widetilde{f}_E,\widetilde{f}) \mapsto \widetilde{f} \mapsto \mathrm{ev}\circ\widetilde{f}$. Let $\overline{f} = \widetilde{f}\circ\pi$ and $S_{\overline{f}} \to \mathbb{P}^1$ be the associated conic bundle.

For $\widetilde{f}$ general in the image of α, we may assume that the source $\widetilde{f}$ is $\mathbb{P}^1$ and if we contract the $3D$ lines in the singular fibers of the conic bundle $S_{\overline{f}} \to \mathbb{P}^1$ not meeting $\widetilde{f}(\mathbb{P}^1)$, we get a surface $\overline{S}$ which is a ruled Hirzebruch surface of type h. The fiber of α over a general point of its image is given by the choice of a section of minimal self intersection of the surface $\overline{S} \to \mathbb{P}^1$ and is therefore of dimension 1 for d even and 0 for d odd. Using our notation, this fiber of α over a general point of its image is of dimension $1-h = 2D+1-d_0$.

We now want to compute the fibers of β. Let $(\widetilde{f}_E,\widetilde{f}) \in \mathcal{M}_{d,m}$ general and set $\overline{f} = \pi\circ\widetilde{f} = \pi\circ\widetilde{f}_E$. We may assume that $\overline{f}$ has irreducible source and let $(p_i)_{i\in[1,m']}$ be the m' marked points on $\mathbb{P}^1$ associated to $\overline{f}$. Set $F_i = \pi^{-1}(p_i)$. Then $C_i = \mathrm{ev}(F_i)$ is a conic in V_5 and set $C_0 = \mathbb{P}^1 \cup \bigcup_{i=1}^{m'} F_i$. Let $\widetilde{f}_0 : C_0 \to M$ be the map defined by $\widetilde{f}_E$ on $\mathbb{P}^1$ and by the inclusion of F_i in M for $i \in [1,m']$. Then $(\widetilde{f}_E,\widetilde{f}_0) \in \mathcal{M}_{d,m}$. Set $f_0 = \mathrm{ev}\circ\widetilde{f}_0$. We compute the dimension of $\mathrm{Hom}(\widetilde{f}_0^*\,\mathrm{ev}^*(\mathbb{C}^5/V_2),\widetilde{f}_0^*\pi^*\mathcal{O}_{F_2(V_5)}(1)) = H^0(C_0,\widetilde{f}_0^*(\mathrm{ev}^*(\mathbb{C}^5/V_2)^\vee\otimes\pi^*\mathcal{O}_{F_2(V_5)}(1)))$. Set $\mathcal{F} = \mathrm{ev}^*(\mathbb{C}^5/V_2)^\vee\otimes\pi^*\mathcal{O}_{F_2(V_5)}(1)$, we want to compute $H^0(C_0,\widetilde{f}_0^*\mathcal{F})$. Let $(q_i)_{i\in[1,m']}$ be the intersection points of $\widetilde{f}_E(\mathbb{P}^1)$ with the fibers $(F_i)_{i\in[1,m']}$. Any global section $s \in H^0(C_0,\widetilde{f}_0^*\mathcal{F})$ is determined by a global section $s_E \in H^0(\widetilde{f}_E^*(\mathbb{P}^1),\widetilde{f}_E^*\mathcal{F})$ and by global section $s_i \in H^0(F_i,\mathcal{F}_{F_i})$ for all $i \in [1,m']$ together with the conditions $s(q_i) = s_i(q_i)$ for all $i \in [1,m']$. Since F_i is a conic (that can be chosen general), we have

$$\mathcal{F}_{F_i} = (\mathcal{O}_{F_i}\oplus\mathcal{O}_{F_i}(1)^2)^\vee\otimes\mathcal{O}_{F_i} = \mathcal{O}_{F_i}\oplus\mathcal{O}_{F_i}(-1)^2$$

so that $\dim H^0(F_i,\mathcal{F}|_{F_i}) = 1$. On the other hand we have

$$\widetilde{f}_E^*\mathcal{F} = \mathcal{O}^1_{\mathbb{P}}(D-a)\oplus\mathcal{O}^1_{\mathbb{P}}(D-b)\oplus\mathcal{O}^1_{\mathbb{P}}(D-c)$$

with $a+b+c=d_E$ and $a\geq b\geq c$. Since $(\widetilde{f}_E,\widetilde{f})$ is general in $\mathcal{M}_{d,m}$ and since $\widetilde{f}_E\in\overline{M}^{\mathrm{bir}}_{0,m'}(M,(D,d_E)))$ we may also assume that $a-c\leq 1$. Our conditions on d and m imply that $D\geq\lfloor\frac{d_E}{3}\rfloor$ thus $D\geq c$ so that $\dim H^0(\mathbb{P}^1,\widetilde{f}_E^*\mathcal{F})=3D-d_E+3$. Our assumptions on d imply $D-a=D-\lceil\frac{d_E}{3}\rceil\geq m'-1$ (see Remark 2.11.(3)). In particular, the linear conditions $s(q_i)=s_i(q_i)$ are linearly independent (see Lemma 2.14) so that $\dim H^0(C_0,\widetilde{f}_0^*\mathcal{F})=\dim H^0(\mathbb{P}^1,\widetilde{f}_E^*\mathcal{F})+\sum_{i=1}^{m'}\dim H^0(F_i,\mathcal{F}|_{F_i}))-3m'=3D-d_E+3-2m'=3D-d+3$. We thus proved that for the point $(\widetilde{f}_E,\widetilde{f}_0)\in\mathcal{M}_{d,m}$, we have $\dim H^0(C_0,\widetilde{f}_0^*\mathcal{F})=3D-d+3$. In particular for any general point $(\widetilde{f}_E,\widetilde{f})\in\overline{M}_{d,m}$ with $\widetilde{f}:C\to M$, we have $\dim H^0(C_0,\widetilde{f}^*\mathcal{F})\leq 3D-d+3$. Since $\mathcal{M}_{d,m}$ is irreducible, so is its image by α. Furthermore this image contains morphisms $\widetilde{f}$ with irreducible source, so these morphisms form a dense subset of this image. In particular, the general fiber of β will contain a dense subset of morphisms $\widetilde{f}$ with irreducible source and for such a morphism, the elements of the fiber are given by an open subset of $\mathbb{P}H^0(\mathbb{P}^1,\widetilde{f}^*\mathcal{F})$ which is of dimension at most $3D-d+2$. The dimension of the general fiber of the composition $\beta\circ\alpha$ is therefore equal to $3D-d+2+2D+1-d_0$. If M_0 is a proper closed subset of $\overline{M}^{\mathrm{bir}}_{0,m'}(V_5,d)$, we get

$$\begin{aligned}\dim\mathcal{M}_{d,m} &< \dim\overline{M}^{\mathrm{bir}}_{m}(V_5,d)+3D-d+2+2D+1-d_0\\ &< 2d+m'+5D-d+3-d_0\\ &\leq d+m'+D+d_E+3=\dim\mathcal{M}_{d,m},\end{aligned}$$

a contradiction. Therefore we have $M_0=\overline{M}^{\mathrm{bir}}_{0,m'}(V_5,d)$. Since $\overline{M}^{\mathrm{bir}}_{0,m'}(M,(D,d))$ is the unique irreducible component of $\overline{M}_{0,m'}(M,(D,d))$ dominating $\overline{M}^{\mathrm{bir}}_{0,m'}(V_5,d)$ this proves (2).

(3) We have seen that the map $\alpha:\mathcal{M}_{d,m}\to\overline{M}^{\mathrm{bir}}_{0,m'}(M,(D,d))$ has fibers of dimension $2D+1-d_0$ so that its image has dimension

$$d+D+m'+d_E+3-(2D+1-d_0)=3D+d+m'+2=\dim\overline{M}^{\mathrm{bir}}_{0,m'}(M,(D,d)),$$

proving the result. □

Remark 2.16. The space $\mathcal{M}_{d,m}$ is even unirational: it is an open subset of a projective bundle over $\overline{M}^{\mathrm{bir}}_{0,m'}(M,(D,d_E))$ which is unirational (see Remark 2.6.(1)).

2.4. **Proof of Theorem 2.1.** Fix $m\geq 0$ and let $d\geq 1$ satisfying the conditions of Hypothesis 2.10.

$d\geq 8m-8$ for d even and $d\geq 8m-3$ for d odd. Recall the definition of d_0, D, d_E and m'.

Lemma 2.17. *For $f\in\overline{M}^{\mathrm{bir}}_{m}(V_5,d)$, there exists $\widetilde{f}\in\overline{M}^{\mathrm{bir}}_{0,m}(F,(D,d))$ with $f=\mathrm{ev}\circ\widetilde{f}$.*

Proof. Since $D\geq\lfloor\frac{d}{3}\rfloor$ (see Remark 2.11.(4)), this follows from Corollary 2.5. □

Lemma 2.18. *Let $f\in\overline{M}^{\mathrm{bir}}_{m}(V_5,d)$ general and $\widetilde{f}\in\overline{M}^{\mathrm{bir}}_{0,m}(F,(D,d))$ general with $f=\mathrm{ev}\circ\widetilde{f}$. In particular assume that both sources of f and $\widetilde{f}$ are $\mathbb{P}^1$ and denote by $(x_i)_{i\in[1,m]}$ the marked points. Set $\overline{f}=\pi\circ\widetilde{f}$.*

If $m \geq 1$, then there exists $\widetilde{f}_0 \in \overline{M}_{0,m}(M,(D,d_0))$ and m fibers $(F_i)_{i\in[1,m]}$ of the map π with $q_i = \widetilde{f}(x_i) \in F_i$ such that setting $\Gamma = \widetilde{f}_0(\mathbb{P}^1)$, setting $C = \Gamma \cup \bigcup_{i=1}^m F_i$ and setting $g = (\mathrm{ev} : C' \to V_5, (q_i)_{i\in[1,m]}) \in \overline{M}_m^{\mathrm{bir}}(V_5, d)$, we have:

(1) *g is a smooth point in the fiber $\mathrm{ev}^{-1}((x_i)_{i\in[1,m]})$ of $\mathrm{ev} : \overline{M}_m^{\mathrm{bir}}(V_5, d) \to V_5^m$.*
(2) *The maps f and g are contained in a $\mathbb{P}^1$ inside $\mathrm{ev}^{-1}((x_i)_{i\in[1,m]})$.*

Proof. Let $\widetilde{f}'$ be obtained from $\widetilde{f}$ by choosing one more marked point for d odd (and $\widetilde{f}' = \widetilde{f}$ for d even). Then $\widetilde{f}' \in \overline{M}_{0,m'}^{\mathrm{bir}}(M,(D,d))$ and by Proposition 2.15, there exists $\widetilde{f}_E \in \overline{M}_{0,m'}^{\mathrm{bir}}(M,(D,d_E))$ such that $(\widetilde{f}_E, \widetilde{f}') \in \mathcal{M}_{d,m}$. Since f and $\widetilde{f}$ are general, so is $\overline{f}$. Furthermore both conditions $D \geq \lfloor \frac{d}{3} \rfloor \geq \lfloor \frac{d_0}{3} \rfloor$ (Remark 2.11.(4)) and $d_0 \geq 2D - 1$ (Remark 2.11.(5)) are satisfied so we may use Proposition 2.8 and Proposition 2.9. In particular by Proposition 2.9 we get the first assertion.

We now prove the second assertion. Consider the conic bundle $\pi : S_{\overline{f}} \to \mathbb{P}^1$ and $E = \widetilde{f}_E(\mathbb{P}^1)$. Let $\overline{S}_{\overline{f},E} \to \mathbb{P}^1$ be the ruled Hirzebruch surface of type h obtained by contracting the lines in the singular fibers of the conic bundles not meeting E. For d even, we set $\widetilde{f}_0 = \widetilde{f}_E$. For d odd, we have $m' = m + 1$ marked points $(x_i)_{i\in[1,m']}$ in $\mathbb{P}^1$ where $x_{m'}$ is the added marked point. Let $F_{m'} = \pi^{-1}(x_{m'})$. The above Hirzebruch surface is of type 1 and a general element in the linear system $|\overline{E}+\overline{F}_{m'}|$ is irreducible. Therefore a general element in the linear system $|E+F_{m'}|$ is irreducible. Let Γ be such an element and $f_0 : \mathbb{P}^1 \to \Gamma$ its (eventual) normalisation.

In both cases (d even or d odd), we have by construction that $\widetilde{f}(\mathbb{P}^1)$ is an element of the linear system $|\Gamma + mF|$ on $S_{\overline{f}}$ and the curves C and $\widetilde{f}(\mathbb{P}^1)$ pass through the m marked points $(\widetilde{f}(x_i))_{i\in[1,m]}$. Any element in the line generated by these elements in the linear system $|\Gamma + mF|$ on $S_{\overline{f}}$ passes through these points. By composition with ev we get a rational family in $\mathrm{ev}^{-1}((x_i)_{i\in[1,m]})$ containing f and g. ☐

For $m \geq 1$, let $\Delta_{2,d-2}$ be the image of the map $\overline{M}_{0,2}^{\mathrm{bir}}(V_5, 2) \times_{V_5} \overline{M}_{0,m}(V_5, d-2) \to \overline{M}_m^{\mathrm{bir}}(V_5, d)$ obtained by gluing the last marked points on each component.

Lemma 2.19. *For $m \geq 1$, if the general fiber $Z' = \mathrm{ev}^{-1}((x_i)_{i\in[1,m-1]})$ of the morphism $\mathrm{ev} : \overline{M}_{0,m}^{\mathrm{bir}}(V_5, d-2) \to V_5^{m-1}$ obtained by evaluating on the first $m-1$ marked points is rationally connected, then the general fiber of the map $\mathrm{ev} : \Delta_{2,d-2} \to V_5^m$ is rationally connected.*

Proof. Let Z be a general fiber of the map $\mathrm{ev} : \Delta_{2,d-2} \to V_5^m$. Forgeting the first component, we have a map $Z \to \overline{M}_{0,m}(V_5, d-2)$. Since any two points lie on a conic in V_5, this map is surjective onto the fiber $Z' = \mathrm{ev}^{-1}((x_i)_{i\in[1,m-1]})$ which is rationally connected. The general fiber of the map $Z \to Z'$ is given by all conic passing through 2 given points. There is a unique such conic so that the map $Z \to Z'$ is birational. ☐

Lemma 2.20. *If $\overline{M}_{0,m}^{\mathrm{bir}}(V_5, d-2)$ is rationally connected, then $\Delta_{2,d-2}$ is rationally connected.*

Proof. We have a map $\overline{M}_{0,2}^{\mathrm{bir}}(V_5, 2) \times_{V_5} \overline{M}_{0,m}^{\mathrm{bir}}(V_5, d-2) \to \overline{M}_{0,m}^{\mathrm{bir}}(V_5, d-2)$. Since any two points lie on a conic in V_5, this map is surjective and the general fiber of the map is given by all 2-pointed conic maping one point to a given point in

V_5. This fiber is therefore rationnally connected and the result follows by the main Theorem in [GHS03]. $\square$

We now prove the following result which is a variant of [DeL15, Lemma 7.9]

Lemma 2.21. *Let $Y \subset Z$ be projective varieties such that*

(1) *Y is rationally connected;*
(2) *for $z \in Z$ general, there exists a $\mathbb{P}^1 \subset W$ containing z and meeting Y along a smooth point of Z.*

Then Z is rationally connected.

Proof. We adapt the proof of [DeL15, Lemma 7.9] and use the existence of the MRC quotient for a strong resolution $\widetilde{Z}$ of Z (see [Kol96]). The MRC quotient is a rational map $\phi : \widetilde{Z} \dashrightarrow Q$ such that a general fiber of the map is an equivalence class for the relation "being connected by a rational curve on $\widetilde{Z}$". By definition, there is some open set U of $\widetilde{Z}$ such that the restriction of ϕ to U is regular, proper, and any rational curve in $\widetilde{Z}$ intersecting U is contained in U. Since the resolution is an isomorphism over the smooth locus of Z, the strict transform of a rational curve through a generic point of Z meeting Y in a smooth point will meet $\widetilde{Y}$ the strict trnsform of Y. By definition of ϕ, this means that $\phi(\widetilde{Y})$ is dense in Q. But since Y is rationally connected we get that Q is a point thus Z is rationally connected. $\square$

To prove Theorem 2.1.(1), it is enough to prove that $\overline{M}_{0,1}^{\mathrm{bir}}(V_5, d)$ is rationally connected since there is a surjective map $\overline{M}_{0,1}^{\mathrm{bir}}(V_5, d) \to \overline{M}_0^{\mathrm{bir}}(V_5, d)$. So we set $m = 1$ and proceed by induction on d. For $d = 1$ or $d = 2$, the result is well know (see Subsection 1.2). We want to apply Lemma 2.21 to $Y = \Delta_{2,d-2}$ and $Z = \overline{M}_0^{\mathrm{bir}}(V_5, d)$. By induction and Lemma 2.20, we obtain that Y is rationally connected and by Lemma 2.18 a general point of z is on a $\mathbb{P}^1$ in Z meeting Y along a smooth point of Z. We can therefore apply Lemma 2.21 and get that Z is rationally connected.

To prove Theorem 2.1.(2), we proceed by induction on m. The result is true for $m = 0$ by Theorem 2.1.(1). We want to apply Lemma 2.21 to Y the general fiber of the map $\mathrm{ev} : \Delta_{2,d-2} \to V_5^m$ and Z the corresponding fiber of the map $\mathrm{ev} : \overline{M}_m^{\mathrm{bir}}(V_5, d) \to V_5^m$. By induction, the general fiber of the map $\mathrm{ev} : \overline{M}_{0,m-1}^{\mathrm{bir}}(V_5, d-2) \to V_5^{m-1}$ is rationally connected. By Lemma 2.19 we obtain that Y is rationally connected and by Lemma 2.18 a general point of z is on a $\mathbb{P}^1$ in Z meeting Y along a smooth point of Z. We can therefore apply Lemma 2.21 and get that Z is rationally connected.

3. Unirationality

In this section we prove the following result.

Theorem 3.1. *Let $d \geq 1$ and $m \geq 0$.*

(1) *For any $d \geq 1$ the moduli space $\overline{M}_{0,0}(V_5, d)$ has pure dimension $2d$. Moreover, for $d \geq 2$, the moduli space $\overline{M}_{0,0}(V_5, d)$ has two irreducible components:*
$$\overline{M}_{0,0}(V_5, d) = \overline{M}_{0,0}^{\mathrm{bir}}(V_5, d) \cup \overline{M}_{0,0}^{\mathrm{line}}(V_5, d).$$
Furthermore,

- $\overline{M}_{0,0}^{\mathrm{bir}}(V_5, d)$ *is unirational;*
- $\overline{M}_{0,0}^{\mathrm{line}}(V_5, d)$ *is rational.*

(2) *For* $m \in \{0, 1\}$ *and* $d \geq 2$*, the general fiber of the evaluation map* $\mathrm{ev} : \overline{M}_{m}^{\mathrm{bir}}(V_5, d) \to V_5^m$ *is unirational.*

Remark 3.2. The statement for $m = 1$ in Theorem 3.1.(2) follows from the same result for $m = 2$ so we will focus on the $m = 2$ case.

3.1. **Preliminary results on $\mathbb{P}^n$ and $\mathcal{Q}_n \subset \mathbb{P}^{n+1}$.** In this subsection we study rational simple connectedness for the projective space and the smooth quadric.

For these varieties, the moduli spaces $\overline{M}_{0,0}(X, d)$ are irreducible (cf. [Tho98], [KP01]). As a consequence the spaces $\overline{M}_{0,0}^{\mathrm{bir}}(X, d)$ are irreducible, so verifying rational simple connectedness reduces to the study of the evaluation morphisms.

Remark 3.3. For these varieties, our notion of rational simple connectedness coincide with rational simple connectedness in [dJS06a]. In particular $\mathbb{P}^n$ and $\mathcal{Q}_n$ are strongly rationally simply connected (cf. [dJS06a, Theorems 1.2]).

Before analysing examples, we should point out that the results on rational simple connectedness we present here for these first examples are well-known. Nonetheless, we push forward the analysis and obtain more precise information on some moduli spaces of rational curves.

3.1.1. *The projective space.* For the projective space $\mathbb{P}^n$ it is easy to obtain the strongest possible statement for rational simple connectedness.

Proposition 3.4. *The projective space* $\mathbb{P}^n$ *is strongly rationally simply connected.*

Before proving the result, let us list some extra properties which hold for the moduli spaces of rational curves on the projective space:

- its VMRT is isomorphic to $\mathbb{P}^{n-1}$;
- the evaluation morphism ev_x from the universal $\mathbb{P}^1$-bundle over the VMRT

$$\begin{tikzcd} \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-1)) \arrow[r, "\mathrm{ev}_x"] \arrow[d, "\pi_x"'] & \mathbb{P}^n \\ \mathbb{P}^{n-1} \arrow[u, bend right, "\sigma_x"'] & \end{tikzcd}$$

coincides with the blow up if $x \in \mathbb{P}^n$.

Let $f \colon \mathbb{P}^1 \to \mathbb{P}^n$ be a degree d morphism, with $d \geq 1$, defined by $[P^0 : \ldots : P^n]$, with $P^i \in \mathbb{C}[u, v]_d$ degree d homogeneous polynomials for $i \in [0, n]$, fix $m = d + 1$ *distinct* points $t_0, \ldots, t_d \in \mathbb{P}^1$ and m arbitrary points $x_0, \ldots, x_d \in \mathbb{P}^n$. We denote by $\mathrm{Mor}_d^{\mathbf{t}\to\mathbf{x}}(\mathbb{P}^1, \mathbb{P}^n)$ the variety of degree d morphisms $f \colon \mathbb{P}^1 \to \mathbb{P}^n$ such that $f(t_i) = x_i$ for all $i \in [0, d]$.

Lemma 3.5. *Keep the notation as above. Then* $\mathrm{Mor}_d^{\mathbf{t}\to\mathbf{x}}(\mathbb{P}^1, \mathbb{P}^n)$ *is rational and isomorphic to*

$$U_d = \{[\lambda_0 : \ldots : \lambda_d] \mid \lambda_i \neq 0 \text{ for all } i \in [0, d]\} \subset \mathbb{P}^d. \tag{3.D}$$

Proof. For $d = 1$, the variety $\mathrm{Mor}_1^{\mathbf{t}\to\mathbf{x}}(\mathbb{P}^1, \mathbb{P}^n)$ coincides with the automorphism group of $\mathbb{P}^1$ with two marked points (since there is a unique line through two points in $\mathbb{P}^n$), which is isomorphic to the multiplicative group $\mathbb{G}_m$ and is then rational.

Assume now $d \geq 2$ and thus $m \geq 3$. In particular, the automorphism group of $\mathbb{P}^1$ preserving m distinct points is trivial. Let $f: \mathbb{P}^1 \to \mathbb{P}^n$ be a degree d morphism and choose coordinates on $\mathbb{P}^1$ such that $t_i = [z_i : 1]$ and choose vectors $v_i \in \mathbb{C}^{n+1}$ such that $x_i = [v_i]$ for all $i \in [0, d]$. In these coordinates we can write $f(t) := f([z : 1]) = [P^0(z) : \ldots : P^n(z)]$. We impose now the conditions $f(t_i) = x_i$ for all $i \in [0, d]$, i.e. that there exist non-zero scalars λ_i, $i \in [0, d]$ such that

$$P(z_i) := (P^0(z_i), \ldots, P^n(z_i)) = \lambda_i v_i.$$

Define

$$L_i(z) := \prod_{j \neq i} \frac{z - z_j}{z_i - z_j},$$

for all $i \in [0, d]$. We therefore have

$$P(z) = \sum_{i=0}^{d} \lambda_i L_i(z) v_i. \tag{3.E}$$

The variety the variety $\mathrm{Mor}_d^{\mathbf{t} \to \mathbf{x}}(\mathbb{P}^1, \mathbb{P}^n)$ is therefore described by (3.D). □

We prove some more results on projective spaces that can be useful for example in the study of rational simple connectedness on del Pezzo surfaces.

We fix $m = d + 1$ and let $m' \geq 0$ be an integer. Let $\mathrm{ev}_{m+m'}$ be the evaluation map

$$\mathrm{ev}_{m+m'} : \overline{M}_{0,m+m'}(\mathbb{P}^n, d) \to (\mathbb{P}^n)^{m+m'}.$$

As in Notation 0.1, we consider the map

$$\Psi_{m+m'} : \overline{M}_{0,m+m'}(\mathbb{P}^n, d) \to \overline{M}_{0,m+m'} \times (\mathbb{P}^n)^{m+m'}.$$

Let $\mathbf{x} = [\mathbf{v}] \in (\mathbb{P}^n)^{m+m'}$ be a fixed point, i.e. we write $x_i = [v_i]$ with $v_i \in \mathbb{C}^{n+1}$ for all $i \in [0, m + m']$. The following lemma provides the numerical condition which guarantees that the general fibre of $\Psi_{m+m'}$ is non-empty and rational. It will be useful to study rationality for moduli spaces of rational curves on del Pezzo surfaces.

Lemma 3.6. *Let $\mathcal{U}_{m+m'} \subset \overline{M}_{0,m+m'}$ the open subset of pairwise distinct points. Assume that the following two conditions hold:*

(1) *$d \geq nm'$;*

(2)

$$\mathrm{rk}\begin{pmatrix} v_0 \wedge v_j & v_1 \wedge v_j & \cdots & v_d \wedge v_j \end{pmatrix}$$
$$= \mathrm{rk}\begin{pmatrix} v_0 \wedge v_j & \cdots & v_{i-1} \wedge v_j & v_{i+1} \wedge v_j & \cdots & v_d \wedge v_j \end{pmatrix}$$

for all $i \in [0, d]$ and $j \in [d + 1, m']$.

Then for any $(\mathbf{t}, \mathbf{x}) \in \mathcal{U}_{m+m'} \times (\mathbb{P}^n)^{m+m'}$ the fibre $\Psi^{-1}_{m+m'}(\mathbf{t}, \mathbf{x})$ is non-empty and rational.

Proof. The case $m' = 0$ is simply Lemma 3.5. More precisely, a morphism $f: \mathbb{P}^1 \to \mathbb{P}^n$ sending the first m pairwise distinct points $(t_0, \ldots, t_d)$ to $(x_0, \ldots, x_d)$ is described by (3.E) and the variety $\mathrm{Mor}_d^{\mathbf{t} \to \mathbf{x}}(\mathbb{P}^1, \mathbb{P}^n)$ is rational of dimension d. Let $(z_{d+1}, \ldots, z_{d+m'})$ be the coordinates of the last m' points in $\mathbb{P}^1$. If we impose the extra condition (i.e. $f(t_j) = x_i$ for all $j \in [d+1, d+m']$), we obtain the following condition on the coefficients $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_d)$:

$$\begin{bmatrix} P^0(z_j) \\ P^1(z_j) \\ \vdots \\ P^n(z_j) \end{bmatrix} = \begin{bmatrix} v_j^0 \\ v_j^1 \\ \vdots \\ v_j^n \end{bmatrix} \tag{3.F}$$

for all $j \in [d+1, d+m']$. These impose at most nm' linear conditions on the λ's. We only need to check that these linear conditions are not of the form $\lambda_i = 0$ for some i, since these would give an empty fibre $\Psi^{-1}_{m+m'}(\mathbf{t}, \mathbf{x})$. Here we need condition (2) in the hypothesis. For any $j \in [d+1, d+m']$, one can develop (3.F) and obtain:

$$A \cdot \boldsymbol{\lambda} := \begin{pmatrix} L_0(z_j)v_0 \wedge v_j & L_1(z_j)v_1 \wedge v_j & \cdots & L_d(z_j)v_d \wedge v_j \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_d \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Condition (2) guarantees that $\mathrm{Ker}(A)$ is not contained in any of the hyperplanes $\lambda_i = 0$, since $L_0(z_j) \neq 0$ for all $j \in [d+1, d+m']$. □

Remark 3.7. The condition (2) of Lemma 3.6 holds if we assume that the points $(x_0, \dots, x_{m+m'})$ are general: indeed, since $d \geq n$, we can choose the first $n+1$ vectors to be the standard basis, i.e. $v_i = e_i$ for $i \in [0, n]$. Since for any $v \in \mathbb{C}^{n+1}$, the matrix $(e_0\ e_1\ \cdots\ e_n) \wedge v$ has rank at most n, the same holds for the matrix $(e_0\ e_1\ \cdots\ e_n\ v_{n+1}\ \cdots\ v_d) \wedge v$. Choosing $v_{n+1}, \dots, v_d$ and v general, condition (2) is verified.

Corollary 3.8. *Let $X := \mathbb{P}^n$ be the n-dimensional projective space, with $n \geq 2$. Then*

$$m - 1 - \left\lfloor \frac{2(m-2)}{n+1} \right\rfloor \leq d_X(m) \leq m - 1 - \left\lfloor \frac{m-1}{n+1} \right\rfloor.$$

Proof. The lower bound is given by dimension count while the upper bound is a direct consequence of Remark 3.7. □

Proof of Proposition 3.4. We know that all moduli spaces $\overline{M}_{0,0}(\mathbb{P}^n, d)$ are irreducible (cf. [FP97, Section 4]). We consider the following diagram:

$$\begin{array}{ccc} \overline{M}_{0,d+1}(X,d) & \xrightarrow{\mathrm{ev}_{d+1}} & X^{d+1} \\ \downarrow{\scriptstyle g} & & \\ \overline{M}_{0,d+1} & & \end{array} \tag{3.G}$$

Lemma 3.5 implies that the general fibre $M_{\mathbf{x}} := \mathrm{ev}^{-1}_{d+1}(\mathbf{x})$ is dominant on $\overline{M}_{0,d+1}$ and that $g_{|M_{\mathbf{x}}}$ is birational to a $\mathbb{P}^d$-fibration over $\overline{M}_{0,d+1}$. Since the base of $g_{|M_{\mathbf{x}}}$ is rational (cf. [Kap93]), we deduce the unirationality of $M_{\mathbf{x}} := \mathrm{ev}^{-1}_{d+1}(\mathbf{x})$. □

3.1.2. *The quadric hypersurface.* Also for this example, we obtain rational simple connectedness in the strongest possible sense. Nonetheless, the proof requires more care, since we need to distinguish two cases, depending on some parity condition.

Proposition 3.9. *The quadric hypersurface $\mathcal{Q}_n \subset \mathbb{P}^{n+1}$, with $n \geq 2$, is strongly rationally simply connected.*

Remark 3.10. The case of $\mathcal{Q}_2 = \mathbb{P}^1 \times \mathbb{P}^1$ is slightly different with respect to the others, since $\rho(\mathcal{Q}_2) = 2$. Nonetheless, we will always fix the diagonal polarisation $\beta = H_1 + H_2$, where the H_i's are the two rulings. With this choice, all the arguments of this section apply also for $n = 2$.

As for $\mathbb{P}^n$ let us recall that, for any $x \in \mathcal{Q}_n \subset \mathbb{P}^{n+1}$, the VMRT is isomorphic to $\mathcal{Q}_{n-2}$, which is then rational, if $n > 2$, or two reduced points if $n = 2$.

As for the previous case, let $f\colon \mathbb{P}^1 \to \mathcal{Q}_n \subset \mathbb{P}^{n+1}$ be a degree d morphism, with $d \geq 1$, defined by $[P^0 : \ldots : P^{n+1}]$, with $P^i \in \mathbb{C}[u,v]_d$ for $i \in [0, n+1]$, fix $m = d+1$ *distinct* points $t_0, \ldots, t_d \in \mathbb{P}^1$ and m arbitrary points $x_0, \ldots, x_d \in \mathcal{Q}_n$. Keeping the analogous notation as before, we denote by $\mathrm{Mor}_d^{\mathbf{t}\to\mathbf{x}}(\mathbb{P}^1, \mathcal{Q}_n)$ the variety of degree d morphisms $f\colon \mathbb{P}^1 \to \mathcal{Q}_n$ such that $f(t_i) = x_i$ for all $i \in [0, d]$.

For this case, we need some more notation. Let V be a $\mathbb{C}$-vector space of dimension $n+2$ and q be a non-degenerate quadratic form on V which defines the quadric $\mathcal{Q}_n \subset \mathbb{P}(V)$, i.e.

$$\mathcal{Q}_n = \{[v] \in \mathbb{P}(V) \mid q(v) = 0\}.$$

Moreover, let B be the non-degenerate bilinear form associated to q.

As in the proof of Lemma 3.5, choose coordinates on $\mathbb{P}^1$ such that $t_i = [z_i : 1]$ and choose vectors $v_i \in V$ such that $x_i = [v_i]$ for all $i \in [0, d]$.

Definition 3.11. Keep the notation as above. The *rescaled skew-symmetric matrix* $A := A_{\mathcal{Q}_n, \mathbf{z}\to\mathbf{v}}$ associated to $(z_0, \ldots, z_d)$ and $(v_0, \ldots, v_d)$ is defined as

$$A_{\mathcal{Q}_n, \mathbf{z}\to\mathbf{v}} := \left(\frac{B(v_i, v_j)}{z_j - z_i} \right)_{i,j \in [0,d]}.$$

The previous definition depends on the choice of coordinates on $\mathbb{P}^1$, but this will not affect our arguments. Let us recall some properties of the Pfaffian of skew-symmetric matrices.

Remark 3.12. Let $A = \{a_{ij}\}$ be an $(m \times m)$ skew-symmetric matrix. The *Pfaffian* $\mathrm{pf}(A)$ of A is defined as

$$\mathrm{pf}(A) := \begin{cases} 0 & \text{if } m \text{ is odd;} \\ \sum_{\sigma \in \mathcal{F}_m} \mathrm{sgn}(\sigma) \prod_{i=1}^k a_{\sigma(2i-1), \sigma(2i)} & \text{if } m = 2k \text{ is even.} \end{cases} \tag{3.H}$$

where $\mathcal{F}_m$ is the set of permutations in $\mathcal{S}_m$ satisfying the following:

- $\sigma(1) < \sigma(3) < \ldots < \sigma(2k-1)$; and
- $\sigma(2i-1) < \sigma(2i)$ for $1 \leq i \leq k$.

One can check that the set $\mathcal{F}_{2k}$ is in 1:1 correspondence with partitions of the set $\{1, 2, \ldots, 2k\}$ into k disjoint subsets with 2 elements:

$$\sigma \in \mathcal{F}_{2k} \xleftrightarrow{1:1} \{(i_1, j_1), \ldots, (i_k, j_k)\}$$

where $i_1 < i_2 < \ldots < i_k$ and $i_l < j_l$ for all $1 \leq l \leq k$. In particular,

$$|\mathcal{F}_{2k}| = \frac{(2k)!}{2^k k!} = (2k-1)!!$$

So, $\mathrm{pf}(A)$ is a degree k polynomial in the entries of A and is linear in its lines and columns. Moreover the following formal properties of $\mathrm{pf}(A)$ hold:

- $\mathrm{pf}(A)^2 = \det(A)$;
- $\mathrm{pf}(BAB^t) = \det(B)\,\mathrm{pf}(A)$ for any $(m \times m)$-matrix B.

Moreover a change of coordinates on $\mathbb{P}^1$ will change the Pfaffian by a non-zero scalar and the vanishing of $\mathrm{pf}(A)$ will not depend on the choice of coordinates. For further details on pfaffians, see [Nor84, Section 5.7].

We can prove an analogue of Proposition 3.4 for quadric hypersurfaces.
Let us recall some known facts. In our notation, given a matrix $A = (a_{i,j})$, the i-th row (resp. the j-th column) is denoted by A_i (resp. by A^j).

Lemma 3.13. *Let $A = (a_{i,j})$ be an $(m \times m)$-skew-symmetric matrix, with indices $i, j \in \{0, 1, \ldots, d\}$. Let $A(i)$ (resp. $A(i,j)$) denote the skew-symmetric matrix obtained from A by removing the i-th row and column (resp. the i-th and the j-th rows and columns).*

(1) *if m is odd and A has rank $m-1$, then $\mathrm{Ker}(A)$ is spanned by the vector $((-1)^1\,\mathrm{pf}(A(0)), \cdots, (-1)^{d+1}\,\mathrm{pf}(A(d)))^t$.*
(2) *If $m = 2k$ is even and A has rank $m-2$, then $\mathrm{Ker}(A)$ is spanned by the m vectors*
$$N_i = ((-1)^{i+j}\,\mathrm{pf}(A(i,j)))_j;$$

Proof. We prove (1): let us define $v_A := ((-1)^1\,\mathrm{pf}(A(0)), \cdots, (-1)^{d+1}\,\mathrm{pf}(A(d)))$. Let $A[i]$ denote the following matrix:
$$A[i] := \begin{pmatrix} A & A^i \\ A_i & 0 \end{pmatrix},$$
which is skew-symmetric of dimension $(m+1) \times (m+1)$, and by construction $\mathrm{pf}(A[i]) = 0$. Using the formal properties of the Pfaffian (see [IW99, Lemma 2.3]), we obtain:
$$0 = \mathrm{pf}(A[i]) = \sum_{j=0}^{d} (-1)^{i+j} a_{i,j}\,\mathrm{pf}(A(j)) = (-1)^{i-1} A_i \cdot v_A^t.$$
In particular v_A^t is in the kernel of A and since A has co-rank one, at least one of the entries of v_A^t is non-zero and so v_A generates the kernel.

To prove part (2), we apply part (1) to deduce that all vectors N_i are in the kernel of A (recall that $\mathrm{pf}(A(i,i)) = 0$). Furthermore, the vectors N_i and N_j are linearly independent as soon as $\mathrm{pf}(A(i,j)) \neq 0$. But the condition on the rank implies that at least one of these Pfaffians is non-zero and that the N_i's span the kernel of A. $\square$

Remark 3.14. The same methods can be used to deduce similar results on the kernel of skew-symmetric matrices for higher co-ranks.

Applying the formal properties of pfaffians to our setting, we deduce the following lemma.

Lemma 3.15. *Keep the notation of Definition 3.11. Let $u, v \in V$ be vectors such that $q(u) = q(v) = 0$ and $B(u,v) = 1$. Let $\mathbf{v} = (v_0, \ldots, v_d)$ be defined via*
$$v_i = \begin{cases} u \text{ for } i \text{ even;} \\ v \text{ for } i \text{ odd} \end{cases}$$
with $i \in [0,d]$. Let $\mathcal{U}_m \subset \overline{M}_{0,m}$ the open subset of pairwise distinct points. Then for any $\mathbf{t} \in \mathcal{U}_m$ (with coordinates $\mathbf{z}$) the matrix $A = A_{\mathcal{Q}_n, \mathbf{z} \to \mathbf{v}}$ has maximal rank.

Proof. The associated matrix A has the form

$$A = \left(\frac{\delta_{i+j\equiv 1}}{z_j - z_i} \right),$$

where $\delta_{i+j\equiv 1}$ equals 1 if $i+j$ is odd and 0 otherwise. For d even, we have $\mathrm{pf}(A) = 0$ while for $d = 2k-1$ odd, we prove the following formula:

$$\mathrm{pf}(A) = \frac{\prod_{i<j,\ i+j\,\mathrm{even}}(z_i - z_j)}{\prod_{i<j,\ i+j\,\mathrm{odd}}(z_i - z_j)}.$$

Indeed, keeping the notation as in Remark 3.12, let $\mathcal{F}_{2k}^{\mathrm{odd}} \subset \mathcal{F}_{2k}$ be the subset of partitions $\sigma = \{(i_1, j_1), \ldots, (i_k, j_k)\}$ such that $i_l + j_l$ is odd for all $l \in [1, k]$. Using formula (3.H), one has:

$$\mathrm{pf}(A) = \sum_{\sigma \in \mathcal{F}_{2k}^{\mathrm{odd}}} \prod_{l=1}^{k} \frac{1}{z_{j_l} - z_{i_l}}. \tag{3.I}$$

Factorising this expression, we deduce:

$$\mathrm{pf}(A) = \frac{P(z_0, z_1, \ldots, z_d)}{\prod_{i<j,\ i+j\,\mathrm{odd}}(z_j - z_i)}.$$

The denominator has degree k^2 and the polynomial $P(z_0, z_1, \ldots, z_d)$ has degree at most $k^2 - k$. If we evaluate P in $z_i = z_j$ for $i < j$ and $i + j$ even, the numerator vanishes, so there exists a constant C for which $P(z_0, z_1, \ldots, z_d) = C \prod_{i<j,\ i+j\,\mathrm{even}}(z_j - z_i)$. One determines the value of $C = 1$ looking at the residue at $(z_0 - z_1)$. □

The following lemma is the quadratic version of Lemma 3.5.

Lemma 3.16. *Keep the notation as above. Then the variety* $\mathrm{Mor}_d^{\mathbf{t}\to\mathbf{x}}(\mathbb{P}^1, \mathcal{Q}_n)$, *with for $d \geq 2$, is either empty or rational and isomorphic to the open subset of $\mathbb{P}^d$ defined by*

$$U_{d,A} = \{[\lambda_0 : \ldots : \lambda_d] \mid \lambda_i \neq 0 \textit{ for all } i \in [0, d] \textit{ and } (\lambda_0, \ldots, \lambda_d) \in \mathrm{Ker}(A)\}. \tag{3.J}$$

Proof. Since $\mathcal{Q}_n$ is contained in the projective space $\mathbb{P}^{n+1}$ we can apply Lemma 3.5 to write down the parametrisation of morphisms in $f\colon \mathbb{P}^1 \to \mathbb{P}^{n+1}$ as in (3.E).
We only need to impose the condition that f factors through $\mathcal{Q}_n$. Let us define

$$Q(z) := q(P(z)) = B(P(z), P(z)). \tag{3.K}$$

By assumption, we have that $F(z_i) = 0$ for all i, so imposing that f factors through $\mathcal{Q}_n$ is equivalent to requiring that Q vanishes with multiplicity at least 2 at the z_i's. Computing the differential $dQ(z) = 2B(dP(z), P(z))$ we have

$$dQ(z) = 2 \sum_{i,j=0}^{d} dL_i(z) L_j(z) \lambda_i \lambda_j B(v_i, v_j) = 2 \sum_{i,j=0,\ i\neq j}^{d} dL_i(z) L_j(z) \lambda_i \lambda_j B(v_i, v_j),$$

where the last equality holds because $B(v_i, v_i) = 0$. An easy computation shows the following formula, for any $l \neq i$:

$$dL_i(z_l) = \frac{1}{z_l - z_i} \frac{\zeta_l}{\zeta_i},$$

where $\zeta_i = \prod_{k\neq i}(z_i - z_k)$. So evaluating the differential $dQ(z)$ in z_l's we obtain:

$$dQ(z_l) = 2 \sum_{i=0,\ i\neq l}^{d} dL_i(z_l)\lambda_l\lambda_i B(v_i, v_l)$$

$$= 2\lambda_l\zeta_l \sum_{i=0,\ i\neq l}^{d} \frac{\lambda_i}{\zeta_i}\frac{B(v_i, v_l)}{z_l - z_i} = 2\lambda_l\zeta_l A_l \cdot \left(\frac{\lambda_0}{\zeta_0}, \cdots, \frac{\lambda_d}{\zeta_d}\right)^t,$$

where A_l is the l-th row of A. So dQ vanishes in the z_l's if and only if the vector $(\lambda_0/\zeta_0, \cdots, \lambda_d/\zeta_d)$ is in $U_{d,A}$. □

Our final aim in this section is to prove rational connectedness of the general fibre of some evaluation morphisms for $\mathcal{Q}_n$: for this, it is enough to look at the spaces $\mathrm{Mor}_d^{\mathbf{t}\to\mathbf{x}}(\mathbb{P}^1, \mathcal{Q}_n)$ for general $\mathbf{z}$ and $\mathbf{x}$ and for sufficiently large d. Nonetheless, in Section 1 we will need some more punctual information about *special* fibres of evaluation maps for $\mathcal{Q}_n$.

Assume that $m = d + 1$ and consider the evaluation map

$$\mathrm{ev}_m : \overline{M}_m^{\mathrm{bir}}(\mathcal{Q}_n, d) \to (\mathcal{Q}_n)^m,$$

as in Notation 0.1. The following lemma describes the general fibres of Ψ_m.

Lemma 3.17. *Keep the notation as above.*

(1) *If d is odd and $\mathbf{t}$ and $\mathbf{x}$ are general,* $\mathrm{Mor}_d^{\mathbf{t}\to\mathbf{x}}(\mathbb{P}^1, \mathcal{Q}_n)$ *is empty;*
(2) *if d is even and $\mathbf{t}$ and $\mathbf{x}$ are general,* $\mathrm{Mor}_d^{\mathbf{t}\to\mathbf{x}}(\mathbb{P}^1, \mathcal{Q}_n)$ *is a point.*

Proof. Let us consider the matrix $A = A_{\mathcal{Q}_n,\mathbf{z}}$, where we chose coordinates $t_i = [z_i : 1]$ on $\mathbb{P}^1$. Then the result follows from Lemma 3.16 and the following claim (in italics).

(1) *if d is odd and $\mathbf{t}$ and $\mathbf{x}$ are general, the matrix A has maximal rank;*
(2) *If d is even and $\mathbf{t}$ and $\mathbf{x}$ are general, the matrix A has rank d and* $\mathrm{Ker}(A)$ *is spanned by a vector with non-zero coordinates.*

Part (1) is a consequence of Lemma 3.15. Let us show Part (2). Since A is of odd dimension $m = d + 1$, it is degenerate. Let $A(0), \cdots, A(d)$ denote the skew-symmetric matrices as in Lemma 3.13. Using Part (1) of the claim, for general $\mathbf{t}$ and $\mathbf{x}$, we have $\mathrm{pf}(A(i)) \neq 0$ for all i. In particular A has rank $d = m - 1$ and, by Lemma 3.13, its kernel is spanned by $((-1)^1 \mathrm{pf}(A(0)), \cdots, (-1)^{d+1} \mathrm{pf}(A(d)))$ which is a vector with non-zero coordinates. This proves (2). □

Corollary 3.18. *Let $X := \mathcal{Q}_n$ be a n-dimensional quadric. Then $d_X(2) = 2$ and for $m \geq 3$,*

$$m - 1 - \left\lfloor \frac{m-3}{n} \right\rfloor \leq d_X(m) \leq m - 1.$$

Proof. The first inequality follows by dimension count. For $m = 2$, it is well known that there is no line but a conic through two general points on $\mathcal{Q}_n$.
Assume $m \geq 3$ and fix m points on $\mathcal{Q}_n$ and let $d = m-1$. If m is odd, Lemma 3.17 gives the result. For even $m \geq 4$, let us define $m' := m - 1$ and $d' := d - 1$. For a general $\mathbf{x}' := (x_1, \ldots, x_{m'}) \in (\mathcal{Q}_n)^{m'}$, again Lemma 3.17 implies that there exists a degree d' morphism $f : \mathbb{P}^1 \to \mathcal{Q}_n$ through $\mathbf{x}'$. Now let x_m be a general point in $\mathcal{Q}_m$. Then the hyperplane H_{x_m} dual to x_m meets $f(\mathbb{P}^1)$ in at least a point y and the line l passing through x_m and y is contained in $\mathcal{Q}_m$ by construction. In particular the

points $(x_1, \dots, x_{m'}, x_m)$ are on the image of a degree d morphism $g\colon \mathbb{P}^1 \cup \mathbb{P}^1 \to \mathcal{Q}_m$ with image $f(\mathbb{P}^1) \cup l$. Since the moduli space of stable maps to $\mathcal{Q}_n$ is irreducible (cf. [KP01, Corollary 1]) and it has a dense open subset of maps from an irreducible curve, we may find a degree d irreducible rational curve passing through general $x_1, \dots, x_m$ in $\mathcal{Q}_n$. □

The following proposition can be seen as a refinement of Lemma 3.17 and will be crucial in Section 1.

Proposition 3.19. *Let γ be an integral curve in $\mathcal{Q}_n$ which is non-degenerate in $\mathbb{P}^{n+1}$, i.e. $\langle\gamma\rangle = \mathbb{P}^{n+1}$. For the product $Z := \gamma^m \subset (\mathcal{Q}_n)^m$, consider the fibre $M := \mathrm{ev}_m^{-1}(Z)$. Then, for any irreducible component M' of M, the image $\Psi_m(M')$ contains a point $(\mathbf{t}, \mathbf{x}) \in \overline{M}_{0,m} \times Z$ such that the corresponding matrix $A = A_{\mathcal{Q}_n, \mathbf{z}\to\mathbf{v}}$, has rank at least $2\left\lceil \frac{d-1}{2} \right\rceil$.*

Proof. First, notice that by Lemma 3.17, the value $2\left\lceil \frac{d-1}{2} \right\rceil$ is the maximal possible rank for the matrix $A_{\mathcal{Q}_n, \mathbf{z}\to\mathbf{v}}$, with $(\mathbf{t}, \mathbf{x}) \in \Psi_m(M)$.
Since the variety $\overline{M}_{0,m}(\mathcal{Q}_n, d) = \overline{M}^{\mathrm{bir}}_{0,m}(\mathcal{Q}_n, d)$ is irreducible of dimension $(n+1)(d+1) - 3$ (cf. [KP01, Corollary 1]) and M is locally defined by $(n-1)(d+1)$ equations, coming from the local equations of γ in $\mathcal{Q}_n$, the irreducible component M' has dimension at least $2d - 1$.

Assume that for a general element $p \in M'$, the matrix $A = A_{\mathcal{Q}_n, \mathbf{z}\to\mathbf{v}}$ associated to $(\mathbf{t}, \mathbf{x}) = \Psi_m(p)$ has rank $2r$.
By Lemma 3.16 the general fibre of $\Psi_m\colon M' \to \Psi_m(M')$ has dimension $d - 2r$ and $\Psi_m(M')$ has dimension at least $d + 2r - 1$. Consider the projection on Z

$$\mathrm{pr}_2\colon \Psi_m(M') \to Z,$$

and let us look at the dimension of its (nonempty) fibres.

Let $K := \{\kappa_1, \dots, \kappa_{2r}\} \subset [0, d]$ with $\kappa_1 < \cdots < \kappa_{2r}$ be indices such that $\mathrm{pf}(A^K_K) \neq 0$ where A^K_K is the matrix obtained form A by the removing the lines and columns of indices outside of K.
Fix $\kappa_0 \in [0, d] \setminus K$ and reorder $[0, d]$ so that $[0, d] = \{\kappa_0, \kappa_1, \dots, \kappa_{2r}, \kappa_{2r+1}, \dots, \kappa_d\}$. Let us define $K_0 := [0, 2r]$ and $K_s := K_0 \cup \{\kappa_s\}$ for any $s \in [2r+1, d]$. By definition, the element $(\mathbf{t}, \mathbf{x})$ satisfies the equations

$$\mathrm{pf}(A^{K_s}_{K_s}) = 0 \text{ for all } s \in [2r+1, d]. \tag{E_s}$$

If $B(v_{\kappa_0}, v_{\kappa_s}) \neq 0$, expanding $\mathrm{pf}(A^{K_s}_{K_s})$ with respect to the line κ_s, we get (cf. the notation of Lemma 3.13 and [IW99, Lemma 2.3]):

$$\begin{aligned} 0 &= \sum_{j=0}^{2r} (-1)^{\kappa_s + j} a_{\kappa_s j} \,\mathrm{pf}(A^{K_0}_{K_0}(j)) \\ &= (-1)^{\kappa_s} a_{\kappa_s \kappa_0} \,\mathrm{pf}(A^K_K) + (-1)^{\kappa_s} \sum_{j=1}^{2r} (-1)^j a_{\kappa_s j} \,\mathrm{pf}(A^{K_0}_{K_0}(j)) \\ &= (-1)^{\kappa_s} \frac{B(v_{\kappa_s}, v_{\kappa_0})}{z_{\kappa_0} - z_{\kappa_s}} \,\mathrm{pf}(A^K_K) + (-1)^{\kappa_s} \sum_{j=1}^{2r} (-1)^j \frac{B(v_{\kappa_s}, v_j)}{z_j - z_{\kappa_s}} \,\mathrm{pf}(A^{K_0}_{K_0}(j)). \end{aligned}$$

So the previous equation is nontrivial and the term $a_{\kappa_s \kappa_0} \,\mathrm{pf}(A^K_K)$ is the only term contributing for a pole along $(z_{\kappa_0} - z_{\kappa_s})$. So all the nontrivial equations (E_s) with

$s \in [2r+1, d]$, are independent, since the equation (E_s) only involves the variables $(z_{\kappa_i})_{i \in K_0}$ and z_{κ_s}.

Let l is the number indices $s \in [2r+1, d]$ such that $B(v_{\kappa_0}, v_{\kappa_s}) = 0$. Since γ is non-degenerate, we get this way $d - 2r - l$ independent equations and hence

$$\dim \mathrm{pr}_2^{-1}(\mathrm{pr}_2(\mathbf{t}, \mathbf{x})) \leq (d-2) - (d - 2r - l) = 2r + l - 2.$$

Up to reordering the indices, we can choose κ_0 so that l is minimal. In particular for each $\tilde{\kappa} \in [0, d] \setminus K$, there are at least l vanishings $B(v_{\tilde{\kappa}}, v_{\kappa_s}) = 0$ for $s \in \{0\} \cup [2r+1, d]$. This in particular implies that we have at least $\frac{l(d+1-2r)}{2}$ equations of the form

$$B(v_i, v_j) = 0 \text{ for } i, j \in [0, d] \setminus K \text{ and } i \neq j.$$

This in turn implies that the dimension of the image $\mathrm{pr}_2(\Psi_m(M'))$ verifies

$$\dim(\mathrm{pr}_2(\Psi_m(M'))) \leq \dim Z - \frac{l(d+1-2r)}{2} = d + 1 - \frac{l(d+1-2r)}{2},$$

and finally that

$$\dim(\Psi_m(M')) \leq d + 1 - \frac{l(d+1-2r)}{2} + 2r + l - 2 = d + 2r - 1 - l\frac{d-1-2r}{2}.$$

Since $\Psi_m(M')$ has dimension at least $d + 2r - 1$, we get $l(d-1-2r) \leq 0$, i.e. either

- $\mathrm{rk}(A) \geq 2\left\lceil \frac{d-1}{2} \right\rceil$; or
- $l = 0$.

We need to treat the second case, so let assume that $l = 0$. The above estimates imply that pr_2 is surjective. We now produce an element $\mathbf{x} \in Z$ such that for any $\mathbf{t} \in \overline{M}_{0,m}$, the corresponding matrix A has maximal rank. Indeed, since γ is non-degenerate, we can find two vectors $u, v \in V$ such that $[u], [v] \in \gamma$ and $B(u, v) = 1$. We conclude applying Lemma 3.15.

We have proved that any component M' of M is such that $\Psi(M')$ contains an elements whose associated matrix A has rank at least $2\left\lceil \frac{d-1}{2} \right\rceil$. □

Proof of Proposition 3.9. Using Corollary 3.18, we see that, for $m = d + 1$ the evaluation maps are dominant. To show that the general fibre $M_{\mathbf{x}} = \mathrm{ev}_m^{-1}(\mathbf{x})$ is rationally connected, we have two cases. If d is even, Lemma 3.17 implies that the map $\psi_m \colon M_{\mathbf{x}} \to \psi_m(M_{\mathbf{x}}) = \overline{M}_{0,m}$ is birational. This implies that $M_{\mathbf{x}}$ is rational (cf. [Kap93]). If d is odd, we use Lemma 3.15 and Lemma 3.16 and deduce that the general fibre of $\psi_m \colon M_{\mathbf{x}} \to \psi_m(M_{\mathbf{x}}) = \overline{M}_{0,m}$ is birational to $\mathbb{P}^1$. Moreover, Lemma 3.13 implies that this morphism has a rational section given by $N_0 = ((-1)^j \,\mathrm{pf}(A(0, j)))_j$. So $M_{\mathbf{x}}$ is rational also in this case. □

3.2. **Proof of Theorem 3.1.** In order to exploit the quasi-homogeneity of V_5, we need few lemmas.

Lemma 3.20. *Let C be a curve in V_5. Then:*

(1) *there exists a line l with $l \cap C = \emptyset$;*
(2) *If C is reduced of degree d and meets the dense orbit U, there exists a line l with $l \cap C = \emptyset$ and such that its intersection with the divisor D_l (spanned by the lines in V_5 meeting l) is a union of d reduced points.*

Proof. Looking at the incidence correspondence (1.B), we notice that he locus of lines meeting C is given by $\varphi(\mathrm{ev}_1^{-1}(C))$, which is a curve in $\mathbb{P}^2$. This proves (1).

For a general line, the surface D_l meets properly the $\mathrm{SL}_2(\mathbb{C})_2$-orbits. By Kleiman-Bertini's theorem (cf. [Kle74, Theorem 2(ii)]), we get that there exists a line l such that $C \cap D_l$ is a finite reduced union of points. $\square$

Lemma 3.21. *Any irreducible component of $\overline{M}_{0,0}(V_5, d)$ contains stable curves whose image meets U, the dense $\mathrm{SL}_2(\mathbb{C})$-orbit.*

Proof. Since any irreducible component of $\overline{M}_{0,0}(V_5, d)$ has dimension at least $2d$, the morphisms that factor through the orbits $E \setminus \sigma$ and σ cannot form irreducible components. Indeed, the subscheme of stable maps that factor through the rational sextic curve σ, is isomorphic to $\overline{M}_{0,0}(\mathbb{P}^1, d/6)$, when d is a multiple of 6 and empty otherwise. In particular is has dimension $d/3 - 2 < 2d$, when non-empty. Recall that the closure E of the two-dimensional orbit is a denormalisation of $\mathbb{P}^1 \times \mathbb{P}^1$ with associated map $\nu\colon \mathbb{P}^1 \times \mathbb{P}^1 \to S$ of degree $(1, 5)$, cf. Lemma 1.2. A similar computation shows that the dimension of stable map that factor through E has dimension $2d - 1 < 2d$ when nonempty: when d is a multiple of 6, this moduli space is given by $\overline{M}_{0,0}(\mathbb{P}^1 \times \mathbb{P}^1, \frac{d}{6}(1, 5))$. So any irreducible component of $\overline{M}_{0,0}(V_5, d)$ contains stable maps whose image meet the dense orbit in V_5. $\square$

The key result is the following.

Proposition 3.22. *For any $d \geq 1$ the moduli space $\overline{M}^{\mathrm{bir}}_{0,0}(V_5, d)$ is irreducible, unirational of dimension $2d$.*

Proof. Since every irreducible component M of $\overline{M}^{\mathrm{bir}}_{0,0}(V_5, d)$ has dimension $\dim M \geq 2d$ (cf. [Kol96, Theorem II.1.2]), let $f\colon \mathbb{P}^1 \to V_5$ be a general element of the dense subset in M. Since $f(\mathbb{P}^1) \cap U \neq \emptyset$, the pull-back $f^* T_{V_5}$ of the tangent bundle of V_5 is globally generated (cf. [Deb01, Example 4.15(2)]). We thus have $H^1(\mathbb{P}^1, f^* T_{V_5}) = 0$ and $\dim H^0(\mathbb{P}^1, f^* T_{V_5}) = 2d$. This proves that $\overline{M}^{\mathrm{bir}}_{0,0}(V_5, d)$ has dimension $2d$.

We now prove that $\overline{M}^{\mathrm{bir}}_{0,0}(V_5, d)$ is irreducible and unirational. Lemma 3.21 implies that there exists a map $[f_0] \in M$ such that $f_0(\mathbb{P}^1)$ meets the dense orbit U and therefore Lemma 3.20(1) implies that there exists a line l such that $l \cap f_0(\mathbb{P}^1) = \emptyset$. In particular, projecting from l as described in Lemma 1.3, we obtain a rational map at the level of moduli spaces:

$$\Phi_l\colon M \dashrightarrow \overline{M}^{\mathrm{bir}}_{0,[d]}(\mathcal{Q}_3, d)$$
$$[f] \mapsto [\phi_l \circ f],$$

where $\overline{M}^{\mathrm{bir}}_{0,[d]}(\mathcal{Q}_3, d)$ is the quotient of $\overline{M}^{\mathrm{bir}}_{0,d}(\mathcal{Q}_3, d)$ by the action of the symmetric group. Since the projection ϕ_l is birational, note that Φ_l is birational onto its image $\Phi_l(M)$. Let us study this image: for a general $f \in M$, its image verifies $f(\mathbb{P}^1) \cap l = \emptyset$, by Lemma 3.20(1) and the image $[\phi_l(f)]$ in $\overline{M}^{\mathrm{bir}}_{0,d}(\mathcal{Q}_3, d)$ is a stable map of degree d. Furthermore $\phi_l(f)$ meets the hyperplane section D_l covered by lines meeting l in d distinct reduced points by Lemma 3.20(2), therefore $\Phi_l(f)(\mathbb{P}^1)$ meets the twisted cubic γ_l in d distinct reduced points.

In particular, let $\mathrm{ev}_d\colon \overline{M}^{\mathrm{bir}}_{0,d}(\mathcal{Q}_3, d) \to \mathcal{Q}_3^d$ be the evaluation map. We have that $\mathrm{ev}_d^{-1}(\gamma_l^d)$ dominates $\Phi_l(M)$, via the quotient by the symmetric group $\mathcal{S}_d$. To prove unirationality of M it is therefore enough to show that $\mathrm{ev}_d^{-1}(\gamma_l^d)$ is unirational of dimension $2d$.

Let $\varphi_{d+1}\colon \overline{M}^{\rm bir}_{0,d+1}(\mathcal{Q}_3,d) \to \overline{M}^{\rm bir}_{0,d}(\mathcal{Q}_3,d)$ the map forgetting the last marked point and $\mathrm{ev}_{d+1}\colon \overline{M}^{\rm bir}_{0,d+1}(\mathcal{Q}_3,d) \to \mathcal{Q}_3^{d+1}$ the evaluation at all marked point. Then $\varphi_{d+1} : \mathrm{ev}_{d+1}^{-1}(\gamma_l^d \times \mathcal{Q}_3) \to \mathrm{ev}_d^{-1}(\gamma_l^d)$ is dominant and since the fibres of the last map have dimension one, it is enough to prove that $\overline{M} := \mathrm{ev}_{d+1}^{-1}(\gamma_l^d \times \mathcal{Q}_3)$ is irreducible and unirational of dimension $2d+1$.

We prove that $\overline{M}$ is actually rational. Let $\overline{M}'$ be an irreducible component of $\overline{M}$; following Notation 0.1, consider the map

$$\Psi_{d+1}\colon \overline{M}' \to \overline{M}_{0,d+1} \times \gamma_l^d \times \mathcal{Q}_3.$$

Since $\overline{M}$ is locally defined by $2d$ equations, coming from the local equations of γ_l in $\mathcal{Q}_3$, the irreducible component $\overline{M}'$ has dimension at least $2d+1$.

Proposition 3.19 implies that $\Psi_{d+1}(\overline{M}')$ contains an elements whose associated matrix A has rank at least $2\left\lceil \frac{d-1}{2}\right\rceil$. We study separately two cases.

- If $d = 2k$ even, a general element in $\Psi_{d+1}(\overline{M}')$ defines a matrix A of rank $2k$ and the fibre is a point, by Lemma 3.17. The map Ψ_{d+1} is therefore birational to its image and has to be dominant since $\overline{M}_{0,d+1} \times \gamma_l^d \times \mathcal{Q}_3$ has dimension $2d+1$. In particular $\overline{M}'$ is rational and there is a unique such irreducible component.
- For $d = 2k+1$ odd, a general element in $\Psi_{d+1}(\overline{M}')$ defines a matrix A of rank $2k$, the fibre is an open subset in $\mathbb{P}^1$ and even an open subset of a $\mathbb{P}^1$ bundle over $\Psi_{d+1}(\overline{M}')$. Indeed, by Lemma 3.13, we have a rational section given by the vector $N_0 = ((-1)^j\,\mathrm{pf}(A(0,j)))_j$. The image $\Psi_{d+1}(\overline{M}')$ is then given by the locus

$$\Psi_{d+1}(\overline{M}') = \{(\mathbf{t},\mathbf{x}) \in \overline{M}_{0,d+1} \times \gamma_l^d \times \mathcal{Q}_3 \mid \mathrm{pf}(A) = 0\}.$$

  Consider the map $\theta\colon \Psi_{d+1}(\overline{M}') \to \overline{M}_{0,d+1} \times \gamma_l^d$ obtained by projection. This map is surjective and its fibre is a linear section of $\mathcal{Q}_3$ therefore a (rational) 2-dimensional quadric. Since θ has a rational section (take $\Psi_{d+1}(\overline{M}') \cap (\overline{M}_{0,d+1} \times (\gamma_l)^d \times L)$ where L is a general line in $\mathcal{Q}_3$), we see that $\overline{M}'$ is rational and there is a unique such irreducible component.

This concludes the proof. □

Proof of Theorem 3.1. We first prove that $\overline{M}^{\rm line}_{0,0}(V_5,d)$ is rational. We have the following isomorphism given by composition of stable maps:

$$\overline{M}_{0,0}(\mathbb{P}^1,d) \times \overline{M}_{0,0}(V_5,1) \to \overline{M}^{\rm line}_{0,0}(V_5,d).$$

It is well known that the moduli space $\overline{M}_{0,0}(\mathbb{P}^1,d)$ is rational of dimension $2d-2$ (cf. [KP01, Corollary 1]) and $\overline{M}^{\rm line}_{0,0}(V_5,1)$ is isomorphic to $\mathbb{P}^2$ so that $\overline{M}^{\rm line}_{0,0}(V_5,d)$ is rational of dimension $2d$.

We prove by induction on d that $\overline{M}_{0,0}(V_5,d)$ has two irreducible components $\overline{M}^{\rm bir}_{0,0}(V_5,d)$ and $\overline{M}^{\rm line}_{0,0}(V_5,d)$, both of dimension $2d$. Indeed, if there exists an irreducible component not contained in $\overline{M}^{\rm bir}_{0,0}(V_5,d) \cup \overline{M}^{\rm line}_{0,0}(V_5,d)$, then it is covered by the images of the gluing maps:

$$\overline{M}_{0,1}(V_5,d_1) \times_{V_5} \overline{M}_{0,1}(V_5,d_2) \to \overline{M}_{0,0}(V_5,d)$$

with $d_1 + d_2 = d$. By induction assumption, this space has dimension at most $(2d_1 + 1) + (2d_2 + 1) - 3 = 2d - 1 < 2d$. Contradiction. So the theorem is a consequence of Proposition 3.22. □

This argument can be easily adapted to obtain rational simple connectedness.

Theorem 3.23. *The Fano threefold V_5 is rationally simply connected.*

Proof. Let $\mathrm{ev}_2 : \overline{M}^{\mathrm{bir}}_{0,2}(V_5, d) \to V_5^2$ be the evaluation map. We want to prove that the general fibre $M_{\mathbf{x}} = \mathrm{ev}_2^{-1}(\mathbf{x})$ is rationally connected of dimension $2d - 4$. We will actually prove that $M_{\mathbf{x}}$ is unirational.

Since $\overline{M}^{\mathrm{bir}}_{0,2}(V_5, d)$ is irreducible of dimension $2d + 2$, any irreducible component M' of $M_{\mathbf{x}}$ contains a curve with smooth source and [Deb01, Proposition 4.14] implies that M' has expected dimension $2d - 4$.

Let l be a line passing through x_1 but not though x_2. By composition with the projection $\phi_l : V_5 \dashrightarrow \mathcal{Q}_3$ (cf. Lemma 1.3 for the notation), a stable map $[f] \in M_{\mathbf{x}}$ with multiplicity one in x_1 is sent to a stable map $\phi_l \circ f$ in $\mathcal{Q}_3$ of degree $d-1$ whose image meets the twisted cubic γ_l in $d-2$ points and which passes through the fibre $\ell := \phi_l^{-1}(x_1)$ over x_1. So, we obtain a rational map at the level of moduli spaces:

$$\Phi_l : M' \dashrightarrow \overline{M}^{\mathrm{bir}}_{0,[d]}(\mathcal{Q}_3, d-1)$$

$$[f] \mapsto [\phi_l \circ f],$$

birational onto its image, Here, $\overline{M}^{\mathrm{bir}}_{0,[d]}(\mathcal{Q}_3, d-1)$ is the quotient of $\overline{M}^{\mathrm{bir}}_{0,d}(\mathcal{Q}_3, d-1)$ by the action of the symmetric group. We look at the image: by construction, ℓ is a line meeting γ_l. Furthermore, since x_2 is in general position, it is outside the indeterminacy locus of ϕ_l, we deduce that the image $\Phi_l(M_{\mathbf{x}})$ is dominated (via the quotient by the symmetric group $\mathcal{S}_d$) by $\mathrm{ev}_d^{-1}(\gamma_l^{d-2} \times \ell \times \{\phi_l(x_2)\})$, where $\mathrm{ev}_d : \overline{M}_{0,d}(\mathcal{Q}_3, d-1) \to \mathcal{Q}_3^d$ is the usual evaluation map.
It is therefore enough to prove that $\overline{M} := \mathrm{ev}_d^{-1}(\gamma_l^{d-2} \times \ell \times \{\phi_l(x_2)\})$ is irreducible and unirational of dimension $2d - 4$.

As in Proposition 3.22 we prove that $\overline{M}$ is actually rational. Let $\overline{M}'$ be an irreducible component of $\overline{M}'$; following Notation 0.1, we look at the map

$$\Psi_d : \overline{M}' \to \overline{M}_{0,d} \times \gamma_l^{d-2} \times L \times \{\phi_l(x_2)\}.$$

The same argument of Proposition 3.22 implies that $\overline{M}'$ has dimension at least $2d - 4$.

Since the points $\mathbf{x} = (x_1, x_2)$ are in general position, we can apply Proposition 3.19 which implies that $\Psi_d(\overline{M}')$ contains an elements whose associated matrix A has rank at least $2\left\lceil \frac{d-2}{2} \right\rceil$. We study separately two cases.

- If $d = 2k + 1$ odd, a general element in $\Psi_d(\overline{M}')$ defines a matrix A of rank $2k$ and we conclude by Lemma 3.17 that Ψ_d is birational onto $\overline{M}_{0,d} \times \gamma_l^{d-2} \times L \times \{\phi_l(x_2)\}$, which is rational.
- For $d = 2k$ even, a general element in $\Psi_d(\overline{M}')$ defines a matrix A of rank $2k - 2$, the fibre is an open subset in $\mathbb{P}^1$ and even an open subset of a $\mathbb{P}^1$ bundle over $\Psi_{d+1}(\overline{M}')$, by Lemma 3.13. We conclude as in Proposition 3.22.

This concludes the proof. □

4. Rationality in low degree

In this final part we give some explicit construction for Hilbert schemes of rational curves in V_5, which imply rationality results for low-degree $\overline{M}^{\mathrm{bir}}_{0,0}(V_5, d)$. These are well-known results, especially up to degree 5. The case $d = 6$ of sextic curves has been studied in [TZ12b] and the authors prove that this space is rational (cf. [TZ12a], [TZ12b, Theorems 1.1-5.1]).

For any integer $d \geq 1$, the Hilbert scheme of degree d rational curves in V_5 is denoted by $\mathcal{H}_{0,d} = \mathcal{H}_{0,d}(V_5)$.

Up to cubic curves, the description of $\mathcal{H}_{0,d}$ is classical.

Lines in V_5. $\mathcal{H}_{0,1} \simeq \mathbb{P}S_2$ (cf. [Isk79, Proposition 1.6(i)], [FN89, Theorem 1]).
Conics in V_5. $\mathcal{H}_{0,2} \simeq \mathbb{P}S_4$ (cf. [Ili94, Proposition 1.22]).
Cubics in V_5. $\mathcal{H}_{0,3} \simeq \mathrm{Gr}(2, S_4)$ (cf. [San14, Proposition 2.46]).

This implies that $\overline{M}^{\mathrm{bir}}_{0,0}(V_5, d)$ is rational for $1 \leq d \leq 3$.

4.1. **Quartics in V_5.** Although the following results are well-known to experts, we provide here geometric proofs.

Proposition 4.1. *The moduli space $\overline{M}^{\mathrm{bir}}_{0,0}(V_5, 4)$ is rational of dimension 8.*

Proof. A rational quartic C in V_5 is clearly non-degenerate, so its linear span is a linear subspace H_C isomorphic to $\mathbb{P}^4 \subset \mathbb{P}^6$. This linear subspace H_C intersects V_5 along a degree 5 curve and, by adjunction formula, this is an elliptic curve. Therefore $H_C \cap V_5$ is the union of C and a line l_C bisecant to C. We therefore obtain a rational map

$$\overline{M}^{\mathrm{bir}}_{0,0}(V_5, 4) \dashrightarrow \overline{M}^{\mathrm{bir}}_{0,0}(V_5, 1).$$

Furthermore, if $l \subset V_5$ is a line and H is a general codimension 2 linear subspace in $\mathbb{P}^6$ with $l \subset H$, then $H \cap V_5$ is the union of l and a degree 4 rational curve. This proves that $\overline{M}^{\mathrm{bir}}_{0,0}(V_5, 4)$ is birational to the variety Z obtained via the following fibre product:

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & \mathrm{Fl}(2,5;7) \arrow[d] \\ \overline{M}^{\mathrm{bir}}_{0,0}(V_5, 1) \arrow[r] & \mathrm{Gr}(2,7), \end{tikzcd}$$

where $\mathrm{Gr}(2;7)$ is the grassmannian of lines in $\mathbb{P}^6$ and $\mathrm{Fl}(2,5;7)$ is the partial flag variety of pairs (l, H) with l a line in $\mathbb{P}^6$ and $H \supset l$ a linear subspace of codimension 2 in $\mathbb{P}^6$. Since the right vertical map is locally trivial in the Zariski topology, the same is true for the left vertical map. The fibres of both vertical maps are rational (isomorphic to $\mathrm{Gr}(3, 5)$) and $\overline{M}^{\mathrm{bir}}_{0,0}(V_5, 1)$ is rational, proving the result. $\square$

We recall some notation on (projective) normality (cf. [SR49, Definition I.4.52]).

Definition 4.2. Let X be an integral variety and let $X \subset \mathbb{P}^n$ be a non-degenerate embedding. One says that X is a *normal subvariety* in $\mathbb{P}^n$ if it is not a projection of a subvariety of the same degree in $\mathbb{P}^N$, with $N > n$.
One says that X is *linearly normal* if the restriction map

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(X, \mathcal{O}_X(1))$$

is surjective.

Remark 4.3. Let $X \subset \mathbb{P}^n$ be non-degenerate. Then one can show the following implications (cf. [Dol12, Proposition 8.1.5]):

- X is a normal subvariety in $\mathbb{P}^n \Rightarrow X$ is linearly normal;
- X is linearly normal and normal $\Rightarrow X$ is a normal subvariety in $\mathbb{P}^n$.

4.2. **Bisecants in V_5.** In order to study moduli spaces of quintic and sextic curves, we need to look at bisecant lines. Let $C \subset V_5$ be a smooth connected curve of degree d and genus g. The set of bisecants to C (i.e. the lines in V_5 meeting C in two points) is expected to be of codimension two in $\mathbb{P}S_2$, that is, finite.

Lemma 4.4. *Keep the notation as above.*
The number of bisecants to C is $\binom{d-2}{2} - 3g$.

Proof. We compute the genus g' of the inverse image C' of C in the incidence correspondence $\overline{M}_{0,1}(V_5, 1)$ (cf. (1.B)) and the degree δ of the projection $C' \to \mathbb{P}S_2$. The projection $\overline{M}_{0,1}(V_5, 1) \to V_5$ is 3-to-1 and ramifies exactly on E (cf. the notation of Lemma 1.2 and [FN89, Lemma 2.3]), which is a section of $\mathcal{O}_{V_5}(2)$, Hurwitz formula implies

$$2g' - 2 = 3(2g - 2) + 2d,$$

giving $g' = 3g + d - 2$. Moreover, $\delta = d$, since the image in V_5 of the inverse image in $\overline{M}_{0,1}(V_5, 1)$ of a line ℓ in $\mathbb{P}S_2$ is the hyperplane section $H_l \cap V_5$ swept out by the lines which intersect the line l, where l is the line of V_5 corresponding in $\mathbb{P}S_2$ to the orthogonal of ℓ. The expected number is the double locus of the projection $C' \to \mathbb{P}S_2$ which is $\binom{d-1}{2} - g'$. □

4.3. **Quintics in V_5 via Desargues configurations.** For any smooth rational quintic C in V_5, one sees it is linearly normal, i.e. it is contained in a unique hyperplane $H_C \subset \mathbb{P}S_6$. Let us consider the surface $S = S_C := H_C \cap V_5$. The surface S is either non-normal, or a (possibly singular) del Pezzo surface of degree 5.

If S is non-normal, then $S = V_5 \cap H_l$, where l is a line in V_5 and H_l is the hyperplane containing the first infinitesimal neighbourhood $\mathrm{Spec}(\mathcal{O}_{V_5}/I^2_{l/V_5})$ of l in V_5, since the non-normal locus is a line (cf. [BS07, Proposition 5.8]). In this case, S is the union of lines in V_5 meeting l (cf. [PS88, Proposition 2.1], [FN89, Corollary 1.3]) and contains a 6-dimensional family of rational normal quintics (excluding the particular case of hyperplane sections of S). For such a quintic, the line l is a triple point of the scheme (of length 3) of bisecant lines to C (cf. Lemma 4.4). Varying the line l, we obtain a 8-dimensional family of quintic curves, which is smaller than the expected dimension 10.

Let us study now the normal case. Let $\tilde{S}$ be the anticanonical model of S, then the class $\mathcal{O}_{\tilde{S}}(C)$ of C in $\tilde{S}$ is given by $\alpha - K_{\tilde{S}}$, for some root $\alpha \in \mathrm{Pic}(\tilde{S})$ (cf. [Dol12, Section 8.2.3]). We recall that the roots of $\mathrm{Pic}(\tilde{S})$, i.e. the orthogonal lattice to $K_{\tilde{S}}$ in $\mathrm{Pic}(\tilde{S})$ (of type A_4) are the differences $\pi_i - \pi_j$, with $i \neq j$, where π_i are the markings of the del Pezzo surface.

One can see the roots in $\mathrm{Pic}(\tilde{S})$ via a self-conjugate (with respect to the fundamental conic) *Desargues configuration* in $\mathbb{P}S_2$. This it a very geometric interpretation of the root system A_4. Assume that S is smooth (i.e. it contains ten lines, which are points in $\mathbb{P}S_2$). The lines of S are the sums $K_S + \pi_i + \pi_j$, indexed by the 2-subsets $\{i, j\} \subset [1, 5]$ and we denote them by ℓ_{ij}. Let M_{ij} be the corresponding point in $\mathbb{P}S_2$. Moreover, two lines ℓ_{ij} and ℓ_{kl} meet if and only if M_{ij} and M_{kl} are conjugate (with respect to the fundamental conic) in $\mathbb{P}S_2$. This is the case if and only if $\{i,j\}\cap\{k,l\}=\emptyset$. So the ten points M_{ij} and their polar lines $M_{ij}^{\perp}$ form a Desargues configuration, i.e. *a line contains three points and a point lies in three lines.* This configuration contains twenty triangles indexed by the roots of $\mathrm{Pic}(\tilde{S})$: a root is an ordered pair (i,j) and we associate to it a triangle in the following way. Take for instance $(1,2)$: we associate to it the triangle $M_{13}M_{14}M_{15}$, which is in perspective from M_{12} to its conjugate $M_{23}M_{24}M_{25}$.

Proposition 4.5. *The moduli space $\overline{M}_{0,0}^{\mathrm{bir}}(V_5,5)$ is rational of dimension 10.*

Proof. Let C be a smooth rational quintic curve in V_5. We can assume it is contained a normal hyperplane section $S=S_C$ of V_5, in fact we may assume the smoothness of S, since the map $C\mapsto S$ is a $\mathbb{P}^4$-fibration: this can be seen computing $H^0(\tilde{S},\alpha-K_{\tilde{S}})=5$ on the canonical model of S.
We know that the class $\mathcal{O}_{\tilde{S}}(C)$ in $\mathrm{Pic}(S)$ is $\alpha-K_S$ for some root α which we choose labeled as $(1,2)$. We obtain:

$$(C\cdot l_{ij})=\begin{cases}0 \text{ if } i=1, j=3,4,5;\\ 2 \text{ if } i=2, j=3,4,5;\\ 1 \text{ otherwise.}\end{cases}$$

the bisecants to C form the triangle $M_{23}M_{24}M_{25}$ (in the Desargues configuration explained above) associated to the root $(2,1)$. Furthermore, we remark that the triangles correspond exactly to the sets of three lines on S generating a hyperplane. We obtained this way a rational map associating to a rational quintic its set of bisecant lines:

$$\begin{aligned}\Psi=\Psi_5\colon \overline{M}_{0,0}^{\mathrm{bir}}(V_5,5) &\dashrightarrow \mathcal{H}\\ C &\mapsto [M_{23},M_{24},M_{25}]\end{aligned}$$

where $\mathcal{H}:=\mathrm{Hilb}(3,\mathbb{P}S_2)$.
We study the fibres of Ψ: choose a general point of $\mathcal{H}:=\mathrm{Hilb}(3,\mathbb{P}S_2)$: the corresponding set τ of three lines in V_5 generates a hyperplane H. Consider the smooth surface $\mathcal{S}:=V_5\cap H$. There exists exactly one root $\alpha\in\mathrm{Pic}(S)$ whose intersection product with the lines corresponding to τ is 1. The quintic rational curves in V_5 whose set of bisecants is τ are exactly the sections of the invertible sheaf $\alpha-K_{\tilde{S}}$ on S and they form a 4-dimensional projective space. Therefore the map Ψ, is (birationally) a $\mathbb{P}^4$-fibration.
We remark that the construction is still meaningful over the field of rational functions of $\mathcal{H}$, i.e. Ψ is Zariski-locally trivial: in fact, the generic point η of $\mathcal{H}$ gives a divisor of a del Pezzo surface S_η defined over $K=\mathbb{C}(\mathcal{H})$, with three lines defined over K. So also the root is defined over K, i.e. $\alpha\in\mathrm{Pic}(S_\eta)$.
To conclude, the variety $\mathcal{H}$ is rational: it is easy to see that $\mathrm{Hilb}(3,\mathbb{P}^1\times\mathbb{P}^1)$ is rational since, once we fix a ruling $\mathbb{P}^1\times\mathbb{P}^1\to\mathbb{P}^1$, there is a birational correspondence between the conics on $\mathbb{P}^1\times\mathbb{P}^1$ and the set of three rules. □

4.4. **Sextics in V_5 via the Segre nodal cubic.** We conclude studying sextic curves in V_5.

Remark 4.6. First, we recall the classical and sporadic isomorphism $\mathrm{Spin}_5\sim\mathrm{Sp}_4$. The group $G=\mathrm{Sp}_4$ has two sets of maximal parabolic subgroups. One of those can be identified with the quadric $\mathcal{Q}_3$ (the closed orbit of the standard representation of G viewed as Spin_5); the other one is the projective space $\mathbb{P}^3$ (the closed orbit of

the standard representation of G viewed as Sp_4).
The variety $\mathcal{B}$ of Borel subgroups of G, viewed inside $\mathcal{Q}_3 \times \mathbb{P}^3$, is the incidence correspondence (line, point), when we view $\mathcal{Q}_3$ as the set of isotropic lines in $\mathbb{P}^3$ with respect to the standard symplectic form. More precisely, the fibre in $\mathcal{B}$ of a point $P \in \mathbb{P}^3$ is a line $\ell_p \subset \mathcal{Q}_3$ and the fibre in $\mathcal{B}$ of a point $Q \in \mathcal{Q}_3$ is an isotropic line $l_Q \subset \mathbb{P}^3$. The relations $Q \in \ell_P$ and $P \in l_Q$ are equivalent.

We provide here a new geometric proof of the following result by Takagi and Zucconi (cf. [TZ12b, Theorem 5.1]).

Proposition 4.7. *The moduli space $\overline{M}_{0,0}^{\mathrm{bir}}(V_5, 6)$ is rational of dimension 12. More precisely, the rational map*

$$\Psi_6 \colon \overline{M}_{0,0}^{\mathrm{bir}}(V_5, 6) \dashrightarrow \mathrm{Hilb}(6, \mathbb{P}S_2)$$

sending a general rational sextic curve to its set of bisecants is birational.

Proof. Lemma 4.4 implies that a rational sextic in V_5 has six bisecant lines, so it remains to prove that for 6 lines in V_5 in general position l_i, with $i \in [0, 5]$, there exists a unique rational sextic which is bisecant to them. Let us fix one of these lines l_0 and consider the projection from it, which we saw in Lemma 1.3:

$$\phi_{l_0} \colon V_5 \dashrightarrow \mathcal{Q}_3$$

which is birational. The rational sextics in V_5 which are bisecant to l_0 are in 1-to-1 correspondence with the rational quartics in $\mathcal{Q}_3$ which are bisecant to γ_{l_0} (keeping the notation of Lemma 1.3). Moreover the strict transforms of the lines $l_1, \dots l_5$ in Q_3 are lines $l_i' \subset \mathcal{Q}_3$ which are secant to γ_{l_0}. Our problem is then reduced to counting rational quartics in $\mathcal{Q}_3$ which are bisecant to the l_i''s and to γ_{l_0}.

The correspondence in Remark 4.6 provides a birational correspondence between the variety parametrising the smooth non-degenerate rational quartics in $\mathcal{Q}_3$, and the variety parametrising the twisted cubics in $\mathbb{P}^3$. Indeed, a rational quartic curve $\delta \in \mathcal{Q}_3$ is the ruling of a quartic scroll $\Sigma_\delta \subset \mathbb{P}^3$ whose double locus is the corresponding twisted cubic $\gamma \subset \mathbb{P}^3$. Conversely, the datum of γ allows to reconstruct δ as the locus of isotropic lines in $\mathbb{P}^3$ which are bisecant to γ.

Moreover, the marked twisted cubic γ_{l_0} corresponds to the ruling of a cubic scroll $\Sigma_{\gamma_{l_0}}$. Since each line $\ell_i \in \mathcal{Q}_3$, with $i \in [1, 5]$ meets γ_{l_0}, the cubic scroll $\Sigma_{\gamma_{l_0}}$ contains the points P_i defined by $\ell_i := \ell_{P_i}$. So our problem is now reduced to enumerating the twisted cubics $\gamma \in \mathbb{P}^3$ through the points $P_i \in \mathbb{P}^3$ and bisecant to two rules $\Sigma_{\gamma_{l_0}}$.

We perform now another birational transformation: let $\mathcal{P} \subset \mathbb{P}^3$ be the union of the five points P_i's and consider the linear system of quadrics passing through $\mathcal{P}$. This defines a birational map

$$\varphi_{\mathcal{P}} \colon \mathbb{P}^3 \dashrightarrow \mathbb{S}_3 \subset \mathbb{P}^4,$$

where $\mathbb{S}_3$ is the 10-nodal Segre cubic primal (cf. [Dol16, Section 2]) and the indeterminacy locus of $\varphi_{\mathcal{P}}$ coincides with $\mathcal{P}$. This map can be seen as the composition

$$\varphi_{\mathcal{P}} = \mathrm{pr} \circ v_2 \colon \mathbb{P}^3 \hookrightarrow v_2(\mathbb{P}^3) \dashrightarrow \mathbb{S}_3 \subset \mathbb{P}^4,$$

where $v_2 \colon \mathbb{P}^3 \to \mathbb{P}^9$ is the second Veronese map and $\mathrm{pr} = \mathrm{pr}_{\langle v_2(\mathcal{P}) \rangle}$ is the projection from the 4-dimensional linear space spanned by $v_2(\mathcal{P})$.

Recall that the system of twisted cubics through $\mathcal{P}$ is transformed into one of the six two-dimensional systems of lines in $\mathbb{S}_3$ (the other five are the transforms of the systems of lines through the p_i's).

The image $v_2(\Sigma_{\gamma_{l_0}})$ is a conic bundle over $\gamma_{l_0} \simeq \mathbb{P}^1$. The normal model of $\Sigma_{\gamma_{l_0}}$ is $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2))$, while the scroll given by union of the planes spanned by the conics (the fibres) of $v_2(\Sigma_{\gamma_{l_0}})$ is $V := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(4))$. One sees that $\deg V = 9$ and $V \cap \langle v_2(\mathcal{P}) \rangle = v_2(\mathcal{P})$. Its projection $W := \mathrm{pr}(V)$ is therefore of degree 4 and so is its dual $W^\vee$, a ruled surface in $(\mathbb{P}^4)^\vee$. By the formula in [Bak60, Chapter IV, Example 2, p. 174], this surface has one singular point T in which two rules meet. The corresponding hyperplane H of $\mathbb{P}^4$ contains two planes on W meeting along a line l_T and containing two conics on the transformed surface $\Sigma_{\gamma_{l_0}}$. The line l_T is the transform of the requested twisted cubic γ. □

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Laboratoire de Mathématiques de Versailles, UVSQ, CNRS, Université Paris-Saclay, 78035 Versailles, France.
E-mail address: andrea.fanelli@uvsq.fr

Laboratoire de Mathématiques de Versailles, UVSQ, CNRS, Université Paris-Saclay, 78035 Versailles, France.
E-mail address: laurent.gruson@uvsq.fr

Laboratoire de Mathématiques de Versailles, UVSQ, CNRS, Université Paris-Saclay, 78035 Versailles, France.
E-mail address: nicolas.perrin@uvsq.fr