ARITHMETIC PROPERTIES OF DEL PEZZO SURFACES

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Written mental notes. These are just some rough notes meant as written mental notes of what the course will cover.

Introduction

In this series of lectures, I will present some results on arithmetic properties of del Pezzo surfaces over a field k. I will try to impose as few restrictions as possible on k. Important fields to keep in mind are

- the complex numbers $\mathbb{C}$;
- algebraically closed fields;
- finite fields $\mathbb{F}_q$;
- p-adic fields $\mathbb{Q}_p$ and their finite extensions;
- number fields.

I will recall and sometimes give proofs of facts on del Pezzo surfaces over algebraically closed fields. I will then take these facts as our guiding principles to figure out what can be said over a field that is not necessarily algebraically closed.

Here is an outline of the course. We begin by studying the *exceptional curves* on del Pezzo surfaces. Next, we focus on del Pezzo surfaces of degree at least 7. The analysis shows that given a k-rational point on a del Pezzo surface, we can often produce more k-rational points. This naturally leads to the question of k-(uni)rationality for del Pezzo surfaces. We will see that del Pezzo surfaces of certain degrees automatically have a k-rational point. Finally, we may touch upon Cox rings of del Pezzo surfaces, universal torsors and Manin's conjecture.

1. Background and the case of curves

Definition 1.1. Let k be a field and let X be a scheme over k. A *k-rational point on X* is a morphism $\operatorname{Spec} k \to X$. We denote the set of all k-rational points of X by $X(k)$.

If $X \subset \mathbb{A}^n_k$ is affine and it is the vanishing set of polynomials $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$, then a k-rational point on X is an n-tuple $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n) \in k^n$ such that $f_1(\overline{x}) = \cdots = f_r(\overline{x}) = 0$.

Lemma 1.2 (Lang-Nishimura)**.** *Let k be a field, let X, Y be varieties over k and let $\varphi\colon X \dashrightarrow Y$ be a rational map. If X has a smooth k-rational point and Y is proper, then Y has a k-rational point.*

Proof. Proceed by induction on the dimension of X. If the dimension of X is 0, then do the obvious. For the induction step, use the valuative criterion of properness to show that a rational map from X to Y determines a rational map from any divisor

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on X to Y. Choose a divisor containing a smooth k-rational point of X as a smooth point and conclude by induction. □

An immediate consequence of the Lang-Nishimura Lemma is that having a k-rational point of a birational invariant among smooth proper varieties.

Corollary 1.3. *If X and Y are smooth proper varieties defined over a field k, then X has a k-rational point if and only if Y has a k-rational point.*

Proof. Use the Lang-Nishimura Lemma and the symmetry of the assumptions. □

Let k be a field and let X be a smooth variety of dimension n defined over k. We denote by ω_X a dualizing sheaf on X and we call it the *canonical line bundle*. Thus, the sheaf ω_X is a line bundle isomorphic to $\bigwedge^n \Omega_X$, the determinant of the sheaf of 1-forms on X. We shall mostly use this for curves and surfaces, i.e. for $n \leq 2$.

Definition 1.4. A *Fano variety over k* is a smooth projective variety X defined over k with ample anti-canonical line bundle $(\omega_X)^\vee$. The degree d_X of X is the positive integer $d_X = (-K_X)^d$.

In this course, we will mostly be interested in the case of Fano varieties of dimension 2, that is, del Pezzo surfaces, over general fields.

In our arguments, we will often switch between a field k and an extension of k, often a fixed algebraic closure $\overline{k}$. In doing so, we will want to keep track of what changes and what stays constant. One of the most basic facts that we will use is the following result.

Fact. Let $k' \supset k$ be an extension of fields. Let X be a projective variety over k and let $\mathscr{L}$ be a line bundle on X. Denote by $\mathscr{L}_{k'}$ the pull-back of $\mathscr{L}$ to the base change $X_{k'}$ of X to k'. The k'-vector space $\mathrm{H}^0(X_{k'}, \mathscr{L}_{k'})$ is isomorphic to the k'-vector space $\mathrm{H}^0(X, \mathscr{L}) \otimes_k k'$.

The more general result with global sections replaced by an arbitrary cohomology group of a quasi-coherent sheaf on X is also true (see Cohomology and base-change). Nevertheless, the mentioned fact is enough for most of our applications. Indeed, we can almost get away simply with knowing that the *dimensions* of the spaces of global sections of line bundles are constant when extending the base field.

Aside. Let us pause to think about "base rings". Suppose that X is the vanishing set of an ideal I in $k[x_1, \ldots, x_n]$. There are at least three "base rings" that can, and will, be relevant for this course.

(*Tautological*) Of course, we defined I to be an ideal in $k[x_1, \ldots, x_n]$, so certainly k is a possible "base ring".

(*Geometric*) We may want to allow ourselves to pick coordinates of points on the intersection of X with curves or surfaces, to extract coefficients of equations of special curves on X, or more generally to be able to make finite extensions of k. Thus, certainly also the collection of all finite extensions of k, i.e. an algebraic closure $\overline{k}$ of k, is a possible "base ring".

(*Minimal*) We may only want to look at the smallest subfield (or even the smallest subring) k_0 of k containing the coefficients of a set of generators of I. Again, k_0 is another good candidate for a "base ring".

Really, we should make up our mind on what the base ring is and stick to the tautological option. For instance, we may be in a situation in which X is defined over the complex numbers by an equation with integer coefficients, and we want to reduce modulo 2 to deduce properties over $\mathbb{C}$. In such a situation, we may (re-)define a surface over $\operatorname{Spec} \mathbb{Z}$ with the same equation as X and we can now base change using $\operatorname{Spec} \mathbb{F}_2 \to \operatorname{Spec} \mathbb{Z}$ or $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{Z}$.

Before looking at the surface case, we give a quick overview of what happens in the case of Fano curves.

If the field k is algebraically closed, then a smooth projective curve over k with ample anti-canonical line bundle (that is, ample tangent bundle) is isomorphic to $\mathbb{P}^1_k$. Thus, we have a full classification, with only one geometric example.

Now, let C be a smooth projective Fano curve over an arbitrary ground field k. The anti-canonical linear system $|\omega_C^\vee|$ on C induces a rational map $C \dashrightarrow |\omega_C^\vee|^\vee$. By the previously mentioned Cohomology and base-change, we know that the linear system $|\omega_C^\vee|$ has dimension 2 and hence we obtain a rational map $\iota\colon C \dashrightarrow \mathbb{P}^2_k$. We fix an algebraic closure $\overline{k}$ of k and base-change to $\overline{k}$. The curve C becomes isomorphic to $\mathbb{P}^1_{\overline{k}}$, the line bundle $|\omega_C^\vee|$ becomes isomorphic to $\mathscr{O}_{\mathbb{P}^1_{\overline{k}}}(2)$ and the rational map ι becomes an isomorphism betwen $\mathbb{P}^1_{\overline{k}}$ and a smooth plane conic. We deduce that ι induces an isomorphism between the curve C and a smooth plane conic.

Observe that $\mathbb{P}^1_k$ is certainly a Fano curve, regardless of whether or not the field k is algebraically closed. The following exercise gives a characterisation of $\mathbb{P}^1_k$ among Fano curves.

Exercise 1.5. Let C be a Fano curve over a field k. The following are equivalent.

(1) The curves C and $\mathbb{P}^1_k$ are isomorphic over k.
(2) There is a k-rational point on C.
(3) There is a line bundle on C of degree 1.
(4) There is a line bundle on C of odd degree.

[Hints. (1) $\Longrightarrow$ (2), use the Lang-Nishimura Lemma.
(2) $\Longrightarrow$ (1), realize C as a conic and use stereographic projection.
(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4), easy.
(3) $\Longrightarrow$ (2), the vanishing set of any global section gives you a point.
(4) $\Longrightarrow$ (3), using a line bundle of odd degree and the canonical line bundle, produce a line bundle of degree 1.]

From the previous exercise, if C is isomorphic to $\mathbb{P}^1_k$ over the field k, then C has at least one k-rational point. We can therefore easily find examples of Fano curves that are not isomorphic to $\mathbb{P}^1_k$.

Exercise 1.6. The conic $C \subset \mathbb{P}^2_{\mathbb{Q}}$ defined by the equation $x^2 + y^2 + z^2 = 0$ is a Fano curve that is not isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$.

We are therefore naturally led to examine the question of whether or not a variety has a rational point. Clearly, if a variety defined over a field k has a k-rational point, then it also has a point over every field extension k' of k. In the case in which the ground field is $\mathbb{Q}$, there are some natural extensions of $\mathbb{Q}$ that play an important role. For instance, the real numbers $\mathbb{R}$ are the completion of $\mathbb{Q}$ with respect to the familiar Euclidean absolute value on $\mathbb{Q}$. Also, for every prime number p, there is a p-adic absolute value on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the p-adic absolute value is denoted by $\mathbb{Q}_p$ and is called the *field of p-adic numbers*. These are the only complete fields containing $\mathbb{Q}$ as a dense subfield (cf Ostrowski's Theorem).

The observation that we made above about k-rational points and extensions applies to the completions of $\mathbb{Q}$. We say that a variety X defined over $\mathbb{Q}$ is *everywhere locally soluble* if the base change of X to $\mathbb{R}$ has an $\mathbb{R}$-rational point and, for every prime p, the base change of X to $\mathbb{Q}_p$ has an $\mathbb{Q}_p$-rational point. Thus, if X has a $\mathbb{Q}$-rational point, then X is everywhere locally soluble. The converse to this statement is usually false, but gives nevertheless an interesting perspective on the set of rational points, in the case of varieties defined over $\mathbb{Q}$. For instance, the Hasse principle holds for a family $\mathscr{F}$ of varieties over $\mathbb{Q}$ if for every variety X in the family $\mathscr{F}$, either X has a $\mathbb{Q}$-rational point, or there is a completion K of $\mathbb{Q}$ such that the base-change $X \times \operatorname{Spec} K$ has no K-rational points. The reason for isolating complete fields is that there is an algorithm for deciding whether varieties over complete fields have rational points or not.

A striking example where the Hasse principle holds is the case of conics. This is a version of the famous Hasse-Minkowski Theorem (see the Wikipedia link for a more general statement).

Theorem 1.7 (Hasse-Minkowski)**.** *Let $C \subset \mathbb{P}^2_{\mathbb{Q}}$ be a smooth conic. The curve C has a $\mathbb{Q}$-rational point if and only if C is everywhere locally soluble. Equivalently, the Hasse principle holds for conics.*

2. del Pezzo surfaces

Let k be a field.

Definition 2.1. A *del Pezzo surface over k* is a smooth projective Fano surface X defined over k.

When the del Pezzo surface X is clear from the context, we often write d, dropping the subscript X, for the degree d_X of X.

Besides the fact that this is a series of lectures at the "Conference/Workshop on *del Pezzo surfaces* and Fano varieties", there is also a somewhat more mathematical motivation for concentrating on del Pezzo surfaces. The Iskovskih-Manin Theorem states that there are two kinds of *minimal* surfaces defined over a field k and such that their base-change to an algebraic closure of k is rational: del Pezzo surfaces and conic bundles. We will not even define conic bundles, although they play a role also in the study of del Pezzo surfaces.

We recall a few facts about del Pezzo surfaces that we will use without proof.

Fact. If the field k is algebraically closed field, then any del Pezzo surface is isomorphic to either $\mathbb{P}^1_k \times \mathbb{P}^1_k$ or to the blow-up of $\mathbb{P}^2_k$ at $r \in \{0, \dots, 8\}$ points in *general position*. The "general position" condition is completely explicit: r points $p_1, \dots, p_r \in \mathbb{P}^2_k$ are in general position if

- $r \leq 8$;
- there is no line in $\mathbb{P}^2_k$ containing three of the points;
- there is no conic in $\mathbb{P}^2_k$ containing six of the points;
- there is no cubic in $\mathbb{P}^2_k$ containing eight of the points that is also singular at one of them.

We will see that the degree of a del Pezzo surface X is a good indication of how complicated X can be. Intuitively, the larger the degree, the easier the surface.

Exercise 2.2. Let k be a field.

(1) Show that $\mathbb{P}^1_k \times \mathbb{P}^1_k$ is a del Pezzo surface.
(2) Show that the blow-up of $\mathbb{P}^2_k$ at r points is a del Pezzo surface if and only if the points are in general position.
(3) Show that the blow-up of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ at one point is isomorphic to the blow-up of $\mathbb{P}^2_k$ at 2 distinct points.
(4) Show that the degree of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ is 8, and the degree of the blow-up of $\mathbb{P}^2_k$ at r points is $9 - r$.

Fact 2.3. Let k be an algebraically closed field and let X be a smooth projective surface over k. The surface X is a del Pezzo surface if and only if either X is isomorphic to $\mathbb{P}^1_k \times \mathbb{P}^1_k$ or there are $r \leq 8$ points in $\mathbb{P}^2_k$ in general position such that X is isomorphic to the blow-up of $\mathbb{P}^2_k$ at these points.

Exercise 2.4. The *Fano plane* is an apt and confusing name for the finite projective plane $\mathbb{P}^2_{\mathbb{F}_2}(\mathbb{F}_2)$ over $\mathbb{F}_2$. Determine the largest number of points in general position in the Fano plane.

If you feel like doing an exercise that I have not tried, for every finite field $\mathbb{F}$, determine the largest number of points in general position in $\mathbb{P}^2_{\mathbb{F}}$.

We denote by K_X a canonical divisor on X. Thus, K_X is an integral linear combination of irreducible divisors on X such that the line bundle associated to K_X is isomorphic to the canonical line bundle $\omega_X \simeq \bigwedge^2 \Omega_X$. Since X is a del Pezzo surface, the divisor $-K_X$ is ample. In fact, more is true.

Fact. Let X be a del Pezzo surface and denote by $d = (K_X)^2$ its degree. The anticanonical linear system $|\omega_X^\vee| = \mathbb{P}\left(\mathrm{H}^0(X, \omega_X^\vee)\right)$ is isomorphic to a projective space of dimension d. Thus, we obtain a rational map $\kappa\colon X \dashrightarrow |\omega_X^\vee|^\vee \simeq \mathbb{P}^d$ which

- is a closed embedding, if the degree of X is at least 3 ($\omega_X^\vee$ is very ample);
- is a morphism, if the degree of X is 2 ($\omega_X^\vee$ is base-point free);
- has a unique base-point, if the degree of X is 1.

Thus, del Pezzo surfaces of degree d at least 3 are (isomorphic to) smooth projective non-degenerate surfaces in $\mathbb{P}^d_k$ of degree d. A del Pezzo surface of degree 2 is the double cover of $\mathbb{P}^2_k$ branched over a plane quartic. A del Pezzo surface of degree 1 is the double cover of the complete intersection of a quadric cone Q in $\mathbb{P}^3_k$ branched over the vertex of the cone Q and the intersection of Q with a cubic surface.

We call the rational map determined by the anti-canonical linear system the *anti-canonical rational map*. Similarly, we will talk about the *anti-canonical embedding* (for degree at least 3) and the *anti-canonical morphism* (for degree at least 2).

Often, we identify the target of the anti-canonical rational map with $\mathbb{P}^d_k$ and we still talk about *the* anti-canonical rational map. In this case, the use of the definite article "the" is imprecise, since it neglects the explicit choice of identification of the dual of the linear system with projective space. Nevertheless, we still prefer the imprecise usage.

Exercise 2.5 (Characteristic not 2)**.** Let k be a field of characteristic different from 2. Let Y, Y' be surfaces over k and let $\pi \colon Y' \to Y$ is a double cover (that is a finite morphism of degree 2). Assume that Y is smooth and Y' is normal. Show that the surface Y' is smooth if and only if branch locus of π is smooth.

As a consequence, every del Pezzo surface of degree 2 over a field of characteristic different from 2 is isomorphic to a double cover of $\mathbb{P}^2_k$ branched over a smooth quartic curve.

Exercise 2.6 (Characteristic 2)**.** Let k be a field of characteristic 2. Show that the branch locus of the anticanonical morphism on a del Pezzo surface of degree 2 defined over k is a double conic. Show that for *any* conic $C \subset \mathbb{P}^2_k$ over k there is a del Pezzo surface for which the branch locus of the anti-canonical morphism is C.

Exercise 2.7. If X is a del Pezzo surface of degree 1, then the blow-up of the base-point of the anticanonical linear system is a rational elliptic surface.

For the following exercise, you will need a little more machinery.

Exercise 2.8. Let X be a del Pezzo surface of degree 1. The linear system associated to $(\omega_X^\vee)^{\otimes 2}$ identifies X with the double cover of a quadric cone Q in $\mathbb{P}^3_k$, branched over the cone vertex of Q and a sextic curve that is the intersection of Q with a cubic surface.

With a view towards being more concrete and to appeal to more projective geometry, we will often talk about anti-canonical divisors $-K_X$ on the del Pezzo surface X. As we have seen, the anti-canonical linear system is never empty and sometimes we may implicitly have an effective anti-canonical divisor in mind, when we talk about $-K_X$. Also, we may sometimes imprecisely talk about *the* anti-canonical divisor, even though there is not necessarily a preferred one, not even among effective ones.

2.1. **Picard group and intersection pairing.** We will see that the Picard group $\operatorname{Pic} X$ encodes a vast amount of geometric and arithmetic information. We briefly recall the definition of the Picard group for a smooth projective variety and, in the case of surfaces, of the intersection pairing on $\operatorname{Pic} X$.

Let X be a smooth projective variety over a field k. Let $\operatorname{Div} X$ denote the free abelian group generated by the classes of the integral divisors in X. The subset of $\operatorname{Div} X$ consisting of all divisors that are linearly equivalent to 0 forms a subgroup of $\operatorname{Div} X$. The quotient of $\operatorname{Div} X$ by the subgroup of divisors linearly equivalent to 0 is the Picard group of X. For del Pezzo surfaces, the group $\operatorname{Pic} X$ is a free finitely generated abelian group. In general, this group has a natural scheme structure. Neglecting the issue of non-reducedness, the connected component of the identity of $\operatorname{Pic} X$ is an abelian variety and the quotient of $\operatorname{Pic} X$ by the connected component of the identity (the *group of components*) is a finitely generated abelian group.

We now restrict to the case of surfaces. The *intersection pairing* is a bilinear pairing $- \cdot - \colon \operatorname{Div} X \times \operatorname{Div} X \to \mathbb{Z}$ with the following properties:

- if $C, D \subset X$ are reduced curves without common components, then $C \cdot D$ is the degree of the scheme-theoretic intersection $C \cap D$;
- if $C, C', D \subset X$ are curves with C, C' linearly equivalent, then the equality
$$C \cdot D = C' \cdot D$$
holds.

The kernel of the intersection pairing contains, by definition, all divisors linearly equivalent to 0. Hence, the intersection pairing descends to a pairing on the Picard group $\operatorname{Pic} X$. We call intersection pairing also the pairing on $\operatorname{Pic} X$ induced by intersection pairing on $\operatorname{Div} X$ and maintain the same symbol. For a general smooth projective surface X, the intersection pairing on $\operatorname{Pic} X$ may still have a non-trivial kernel. Nevertheless, for a del Pezzo surface the intersection pairing on $\operatorname{Pic} X$ is non-degenerate.

Let $C \subset X$ be a reduced curve and denote by $p_a(C)$ the arithmetic genus of C. A useful property of the intersection pairing is the formula $C^2 + C \cdot K_X = 2p_a(C) - 2$. This identity is often called the *adjunction formula*.

I would like to make explicit what is the effect of changing the ground field on $\operatorname{Pic} X$ and on the intersection pairing. Let X be a smooth projective surface defined over a field k and let $k' \supset k$ be a field extension. Denote by X' the base change of X to $\operatorname{Spec} k'$. Clearly, every curve C on X (this implicitly implies that the curve is defined over k), also determines, by base change, a curve C' on X'. This produces a natural homomorphism $\operatorname{Pic} X \to \operatorname{Pic} X'$ that is injective (see, for instance, the Stacks Project Lemma 32.30.3). Naturally, even if the curve C is irreducible, the curve C' need not be. This is not a problem for the Picard group, but it highlights a potential source of differences between $\operatorname{Pic} X$ and $\operatorname{Pic} X'$: the classes of the irreducible components of C' need not be in the image of $\operatorname{Pic} X \to \operatorname{Pic} X'$.

Exercise 2.9. Let $C \subset \mathbb{P}^2_{\mathbb{R}}$ be the conic curve with equation $x^2 + y^2 + z^2 = 0$. The Picard group of C is isomorphic to $\mathbb{Z}$, generated, for instance, by the class of a section of C by a line. The inclusion $\operatorname{Pic} C \to \operatorname{Pic} C_{\mathbb{C}}$ is not surjective: what is the index of $\operatorname{Pic} C$ in $\operatorname{Pic} C_{\mathbb{C}}$?

Exercise 2.10. Let $a \in \mathbb{R} \setminus \{0\}$ be a non-zero real number and let $Q_a \subset \mathbb{P}^3_{\mathbb{R}}$ be the smooth quadric surface with equation $x^2 + y^2 + az^2 = w^2$.

- Compute the Picard group of the base change $(Q_a)_{\mathbb{C}}$ of Q_a to $\mathbb{C}$.
- The Picard group of Q_1 is isomorphic to $\mathbb{Z}$, generated by the class of a plane section.
- The Picard group of Q_{-1} is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, generated by the two rulings on Q_{-1}.
- What is the Picard group of Q_a, for the remaining values of a? Describe the inclusion $\operatorname{Pic} Q_a \to \operatorname{Pic}(Q_a)_{\mathbb{C}}$.
- How do your answers change if we choose a to be a rational number and we define Q_a over $\mathbb{Q}$ instead?

We have therefore seen examples with the inclusion $\operatorname{Pic} X \to \operatorname{Pic} X'$ is not primitive, is not of finite index, is an isomorphism. If you did not already do so, it may be useful to think about whether and how the presence of rational points interacts with your answers. In particular, when there are points, you can define divisors by considering intersections with the tangent space at rational points.

2.2. **Exceptional curves.** Using the intersection pairing, we define the curves that will keep us busy for quite a bit.

Definition 2.11. Let X be a smooth projective surface over a field k. An *exceptional curve on* X is a smooth projective curve $E \subset X$ such that $E \cdot K_X = E^2 = -1$.

A consequence of this definition and the adjunction formula is that the arithmetic genus of an exceptional curve is 0: the curve E is a Fano curve! Moreover, the restriction to E of the canonical line bundle on X is a line bundle on E of degree -1. We saw in Exercise 1.5 that E is therefore isomorphic to $\mathbb{P}^1_k$ over k.

Of course, a smooth projective surface need not contain any exceptional curve, even if it is a del Pezzo surface. For instance, neither $\mathbb{P}^2_k$ nor $\mathbb{P}^1_k \times \mathbb{P}^1_k$ contain any exceptional curves. Nevertheless, at least over an algebraically closed field, these are the only del Pezzo surfaces that do not contain an exceptional curve. Over a general field k, it may happen that there are exceptional curves defined over $\overline{k}$, but that none of these are defined over k. Indeed, for instance over number fields, there are ways of quantifying this statement (Hilbert Irreducibility Theorem and thin sets) and deduce that this usually the case.

Fact 2.12. Let X be a smooth surface over a field k and let $E \subset X$ be an exceptional curve. There is a smooth surface X' over k with a point $p \in X(k)$ and a proper birational morphism $\pi \colon X \to X'$ such that

- the morphism π is an isomorphism away from p;
- the fiber of π above p is the exceptional curve E;
- the formula $K_X = \pi^* K_{X'} + E$ holds;
- if X is projective, then $(K_{X'})^2 = (K_X)^2 + 1$.
- if X is a del Pezzo surface, then X' is a del Pezzo surface.

These facts justify our interest in exceptional curves. First, exceptional curves are precisely the result of a blow-up and can therefore be blown down. Second, the square of the canonical divisor increases by 1 in a blow-down. Thus, if X is a del Pezzo surface of degree d over field k and it contains an exceptional curve, then we can blow down the exceptional curve and obtain a del Pezzo surface of higher degree (and hence likely easier).

We begin by analyzing the exceptional curves on blow-ups of $\mathbb{P}^2_k$ at $r \leq 8$ general k-rational points. By Fact 2.3, these are del Pezzo surfaces and conversely, over an algebraically closed field, every del Pezzo surface is of this form, up to isomorphism. As we mentioned, there are no exceptional curves on $\mathbb{P}^2_k$.

> **Aside.** Here is a fun example to think about, that plays with the issue of del Pezzo surfaces and blow-ups of $\mathbb{P}^2_k$. The Fermat equation $x^3 + y^3 + z^3 + w^3 = 0$ describes a smooth cubic surface F in $\mathbb{P}^3_{\mathbb{F}_2}$. Thus, F is a del Pezzo surface of degree 3 over $\mathbb{F}_2$ and as such, it is the blow-up of $\mathbb{P}^2_{\overline{\mathbb{F}}_2}$ at 6 points in general position. Nevertheless, we saw in Exercise 2.4 that the Fano plane does not have 6 points in general position. We conclude that the Fermat surface F over $\mathbb{F}_2$ is **not** the blow-up of $\mathbb{P}^2_{\mathbb{F}_2}$ at 6 points in general position. Of course, any smooth cubic surface, or even any del Pezzo surface of degree at most 4, over $\mathbb{F}_2$ would highlight the same point.

Let $r \leq 8$ be a non-negative integer, let $p_1, \ldots, p_r$ be k-rational points of $\mathbb{P}^2_k$ in general position and let X_r denote the blow-up of $\mathbb{P}^2_k$ at these r points. The notation for X_r already suggests that the specific location of the blown up points will not play a role in our considerations: this is true, even though the surfaces obtained by blowing up $\mathbb{P}^2_k$ at different choices of points may be non-isomorphic.

By construction, the exceptional divisors $E_1, \dots, E_r$ over the points $p_1, \dots, p_r$ are exceptional curves on X_r.

Exercise 2.13. The Picard group of X_r is isomorphic to $\mathbb{Z}^{r+1}$. A basis of $\operatorname{Pic} X_r$ consists of the class ℓ of the inverse image of a line in $\mathbb{P}^2_k$ and the classes $e_1, \dots, e_r$ of the r exceptional divisors $E_1, \dots, E_r$ over the blown-up points. The intersection pairing among the elements of this basis is

$$\ell \cdot \ell = 1; \qquad \ell \cdot e_1 = \ell \cdot e_r = 0; \qquad e_i \cdot e_j = \begin{cases} -1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

Let $\ell, e_1, \dots, e_r$ be the basis of $\operatorname{Pic} X_r$ constructed in the previous exercise, so that every divisor class on X_r is of the form

$$a\ell - \sum_{i=1}^{r} a_i e_i, \tag{1}$$

with $a, a_1, \dots, a_r$ integers. Note the slightly unusual choice of negative signs in front of the coefficients $a_1, \dots, a_r$: this choice is of course immaterial, but, as a partial justification, the coefficients of $e_1, \dots, e_r$ are negative for classes of integral curves different from $e_1, \dots, e_r$. To simplify the notation, we write $(a; a_1, \dots, a_r)$ for the class (1).

Exercise 2.14. The class of the anticanonical divisor on X_r is

$$-K_{X_r} = 3\ell - (e_1 + \cdots + e_r) = (3; 1, \dots, 1).$$

We are now ready to find all the exceptional curves on X_r.

Exercise 2.15. Table 1 lists the classes $e \in \operatorname{Pic} X_8$ satisfying the equations

$$e^2 = -1 \qquad e \cdot K_{X_8} = -1,$$

up to permutation of the coordinates $a_1, \dots, a_8$. To obtain the classes on a del Pezzo surface X_r when $r \leq 7$, simply drop the appropriate number of coordinates equal to 0.

(0 ; -1 , 0 , 0 , 0 , 0 , 0 , 0 , 0)
(1 ; 1 , 1 , 0 , 0 , 0 , 0 , 0 , 0)
(2 ; 1 , 1 , 1 , 1 , 1 , 0 , 0 , 0)
(3 ; 2 , 1 , 1 , 1 , 1 , 1 , 1 , 0)
(4 ; 2 , 2 , 2 , 1 , 1 , 1 , 1 , 1)
(5 ; 2 , 2 , 2 , 2 , 2 , 2 , 1 , 1)
(6 ; 3 , 2 , 2 , 2 , 2 , 2 , 2 , 2)

TABLE 1. Exceptional curves on the del Pezzo surface X_r

By definition, the class of an exceptional curve on X_r must appear in Table 1. All that we are missing now is to know that these classes really do represent exceptional curves.

Exercise 2.16. If $e \in \operatorname{Pic} X_r$ is a class satisfying $e^2 = e \cdot K_{X_r} = -1$, then e is the class of a unique exceptional curve on X_r.
[Hint. Use Riemann-Roch and Serre duality to show that e is the class of a single curve on X_r. Then, use the ampleness of $-K_{X_r}$ to deduce that this curve is irreducible and hence smooth.]

We summarize what we proved in the following Theorem.

Theorem 2.17. *Let k be a field and let X_r be the blow-up of $\mathbb{P}^2_k$ at $r \leq 8$ points in general position. The exceptional curves on X_r are the strict transforms of*

(1) lines containing two of the blown up points ($r \geq 2$);
(2) conics containing five of the blown up points ($r \geq 5$);
(3) cubics containing seven blown up points and singular at one of them ($r \geq 7$);
(4) quartics containing the eight blown up points and singular at three of them ($r = 8$);
(5) quintics containing the eight blown up points and singular at six of them ($r = 8$);
(6) sextics singular at all eight blown up points and with a triple point at one of them ($r = 8$).

r	Number of exceptional curves	
0	0	$= 0$
1	1	$= 1$
2	3	$= 2 + \binom{2}{2}$
3	6	$= 3 + \binom{3}{2}$
4	10	$= 4 + \binom{4}{2}$
5	16	$= 5 + \binom{5}{2} + \binom{5}{5}$
6	27	$= 6 + \binom{6}{2} + \binom{6}{5}$
7	56	$= 7 + \binom{7}{2} + \binom{7}{5} + \binom{7}{1}$
8	240	$= 8 + \binom{8}{2} + \binom{8}{5} + 8 \cdot 7 + \binom{8}{3} + \binom{8}{6} + \binom{8}{1}$

TABLE 2. Exceptional curves on the del Pezzo surface X_r

Theorem 2.17 gives the list of exceptional curves on X_r over k. Because the list is independent of k, there are no further exceptional curves on X_r, even allowing for an extension of the ground field.

Before we move on exceptional curves to del Pezzo surfaces that are not necessarily the blow-up of $\mathbb{P}^2_k$ over the ground field k, we introduce one more tool.

Definition 2.18. The *graph of exceptional curves on X_r* is the finite, undirected, loopless (multi-)graph with vertices indexed by the exceptional curves on X_r and with $e \cdot f$ edges joining the vertices corresponding to the exceptional curves e, f on X_r.

Let Γ_r be the graph of exceptional curves on X_r. Observe that Γ_r is simple (that is, it does not have multiple edges) if $r \leq 6$. The multi-graph Γ_7 has 28 pairs of vertices joined by double edges. The multi-graph Γ_8 has 6720 pairs of vertices joined by double edges and 120 pairs of vertices joined by triple edges.

Exercise 2.19. Show that

(1) Γ_2 is a tree with 3 vertices;
(2) Γ_3 is a hexagon;
(3) Γ_4 is the Petersen graph.

The remaining simple graphs of exceptional curves also have names: Γ_5 is the (5-regular) Clebsch graph and Γ_6 is the Schläfli graph. The complement of the Gosset graph is the graph obtained from the multi-graph Γ_7 by replacing each double edge with a simple edge. The multi-graph Γ_8 may or may not be related to the Gosset polytope with Coxeter symbol 4_{21}. I did not check this, but you can if you want to: if you do, then let me know!

We now turn our attention to an arbitrary del Pezzo surface X of degree d over a field k. By "arbitrary", we mean that we do not require it to be a blow-up of $\mathbb{P}^2_k$. Our initial goal is to understand over which extensions of k the del Pezzo surface X is isomorphic to the blow-up of $\mathbb{P}^2_k$.

Definition 2.20. Let X be a del Pezzo surface X over a field k and let $\overline{X}$ be the base change of X to the algebraic closure $\overline{k}$ of k. The surface X is *split* if the inclusion $\operatorname{Pic} X \to \operatorname{Pic} \overline{X}$ is an isomorphism.

Exercise 2.21. Show that a del Pezzo surface X of degree d over a field k is split if and only if either X is isomorphic to $\mathbb{P}^1_k \times \mathbb{P}^1_k$, or X is isomorphic over k to the blow-up of $\mathbb{P}^2_k$ at $9-d$ points in general position.

Fix an algebraic closure $\overline{k}$ of k and denote by $\overline{X}$ the base change of X to $\overline{k}$. Regardless of whether or not X is split, the classes of the exceptional curves on X inject, under the inclusion $\operatorname{Pic} X \to \operatorname{Pic} \overline{X}$, in the set of classes of exceptional curves on $\overline{X}$. Let $\operatorname{Gal}(k)$ denote the absolute Galois group of k, that is the group of all field automorphisms of $\overline{k}$ that restrict to the identity on k. Since X is defined over k, the elements of the Galois group $\operatorname{Gal} k$ send any subscheme $Y \subset \overline{X}$ to a subscheme of $\overline{X}$. It is easy to convince yourself that this defines an action of $\operatorname{Gal}(k)$ on $\operatorname{Pic} \overline{X}$ that preserves the intersection pairing. By construction, the image of $\operatorname{Pic} X$ in $\operatorname{Pic} \overline{X}$ is fixed by the action of the Galois group. In particular, the canonical divisor class in $\operatorname{Pic} \overline{X}$ is fixed by the action of $\operatorname{Gal}(k)$.

Exercise 2.22. Give an example of a smooth projective variety Z defined over a field k and a divisor class $d \in \operatorname{Pic} \overline{Z}$ such that d is fixed by the action of the Galois group $\operatorname{Gal}(k)$ and d is not in the image of $\operatorname{Pic} Z$.
[Hint. You can choose Z to be a Fano curve.]

Let $\operatorname{Isom}(X) = \operatorname{Isom}(\operatorname{Pic} \overline{X}, K_X)$ be the group of automorphisms of the Picard group of $\overline{X}$ that preserve the intersection pairing and fix the class of the canonical divisor. From the previous discussion, we deduce that there is a homomorphism

$$\operatorname{Gal}(k) \to \operatorname{Isom}(\operatorname{Pic} \overline{X}, K_X).$$

Exercise 2.23. In this exercise, we show that the group $\operatorname{Isom}(X)$ is finite.

- Compute $\operatorname{Isom}(\mathbb{P}^1_k \times \mathbb{P}^1_k)$.

- If X is the blow-up of $\mathbb{P}^2_k$ at $r \leq 8$ points in general position, then show that $\operatorname{Isom}(X)$ is isomorphic to the automorphism group of the graph Γ_r.
- Deduce that if X is a del Pezzo surface of degree 1, then the order of $\operatorname{Isom}(X)$ is $696729600 = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$.

[Hints. When do the exceptional curves generate $\operatorname{Pic} \overline{X}$?
$696729600 = 240 \cdot 56 \cdot 27 \cdot 16 \cdot 10 \cdot 6 \cdot 2 \cdot 1$.]

Lemma 2.24. *If X is a del Pezzo surface over a field k and $e \in \operatorname{Pic} \overline{X}$ is the class of an exceptional curve that is fixed by the action of the Galois group* $\operatorname{Gal}(k)$*, then e is the divisor class of an exceptional curve on X.*

Proof. Let $d = (K_X)^2$ be the degree of X. We will only argue this if

- the anti-canonical linear system is very ample (that is, $d \geq 3$); and
- the field k is perfect.

Neither of these assumptions is necessary, but they will allow us to give an argument using classical projective geometry.

Using the anti-canonical embedding, X maps to a surface in $\mathbb{P}^d_k$ and every exceptional curve on $\overline{X}$ maps to a line. Thus, the Fano scheme $F(X)$ of lines on X is a finite subscheme of the Grassmannian of lines in $\mathbb{P}^d_k$. The $\overline{k}$-point η of $F(X)$ corresponding to the exceptional divisor e is therefore fixed by the Galois group. It follows that the Plücker coordinates of η fixed by the Galois group are hence contained in a purely inseparable extension of k. By the assumption that the field is perfect, we deduce that η is a k-rational point of the Grassmannian. We conclude that there is a line defined over k corresponding to e and we are done. $\square$

Aside. In the proof of the previous lemma, we made two simplifying assumptions. First, we used that the exceptional curves on $\overline{X}$ could be viewed as lines under the anti-canonical embedding to parametrize them using a Grassmannian. When the anti-canonical linear system is not ample, then we can replace it by a very ample multiple and use an appropriate Hilbert scheme, instead of the Grassmannian. Second, we used the assumption on the ground field being perfect to deduce that the coordinates of the point corresponding to e were contained in the ground field, since they were fixed by the Galois group. With a bit of thought, we have constructed a zero-dimensional scheme Z over k (in a Grassmannian or a Hilbert scheme) whose support consist of a single point (corresponding to e). This is very close to having found a k-rational point: all that we are missing is the information that the degree of Z is 1. If we know that the scheme Z is **smooth**, then we can conclude. This is true for exceptional curves on del Pezzo surfaces, since their "deformations are unobstructed".

Exercise 2.25. Find an example of a scheme Z over a field k and an extension $k' \supset k$ such that

- Z is reduced;
- $Z \times_{\operatorname{Spec} k} \operatorname{Spec} k'$ is not reduced.

[Hint. You can choose Z to be zero-dimensional.]

There is also a related fact, that we will not need, but we state it, because of its usefulness in practice.

Fact 2.26. Let X be a smooth projective variety defined over a field k and let $\overline{k}$ be an algebraic closure of k. If $X(k)$ is not empty, then the inclusion $\operatorname{Pic} X \to \operatorname{Pic} \overline{X}$ is an isomorphism.

To structure our approach, we introduce the following notion.

Definition 2.27. A variety X over a field k is *k-rational* if X and $\mathbb{P}_k^{\dim X}$ are birational using a map defined over k. A variety X over a field k is k-unirational if there is an integer n and a dominant rational map $\mathbb{P}_k^n \dashrightarrow X$ defined over k.

Theorem 2.28. *A del Pezzo surface X of degree 9 over a field k is k-rational if and only if X has a k-rational point.*

Sketch of proof. One direction is clear: regardless of what the field k is, the projective plane $\mathbb{P}_k^2$ always has k-rational points.

Suppose that X is a del Pezzo surface of degree 9 with a k-rational point p. First, we show that there is at least one more point on X. Indeed, the anti-canonical linear system is a 9-dimensional projective space of plane cubics defined over k. The linear subsystem corresponding cubics singular at p is defined over k and has dimension 7. Let C be a geometrically irreducible element of this linear system (is there one such element? What happens if k is finite?). The normalization $\underline{C}$ of C is therefore a Fano curve with a line bundle of odd degree. Thus $\underline{C}$ is isomorphic to $\mathbb{P}_k^1$, and hence C has k-rational points away from p (again, what happens if k is finite?). We deduce that there is a k-rational point q on X different from p.

Under the anti-canonical linear system, the surface X embeds in $\mathbb{P}_k^9$. Denote by H_X the closed subscheme of the Hilbert scheme of twisted cubics in $\mathbb{P}_k^9$ that are contained in the image of X. The scheme H_X is defined over k, since X is, and it is 3-dimensional, since, over $\overline{k}$ it corresponds to the linear system of lines in $\overline{X} \simeq \mathbb{P}_{\overline{k}}^2$. Moreover, the closed subscheme of H_X consisting of twisted cubics containing both p and q is zero-dimensional and of degree 1 (check smoothness!): we just proved that the line $L \subset \overline{X}$ through p and q is defined over k. We conclude that X is isomorphic to $\mathbb{P}_k^2$, using the morphism associated to the linear system $|L|$. □

> **Aside.** Forms of projective space over k, that is Brauer-Severi varieties can also be studied using Brauer groups. We will not pursue this direction.

We now give a few easy examples of how we can use the information on $\mathrm{Isom}(X)$.

Theorem 2.29. *Let X be a del Pezzo surface of degree 8. Either $\overline{X}$ is isomorphic to $\mathbb{P}_{\overline{k}}^1 \times \mathbb{P}_{\overline{k}}^1$, or X is isomorphic over k to the blow-up of $\mathbb{P}_k^2$ at one k-rational point.*

Proof. If $\overline{X}$ is isomorphic to $\mathbb{P}_{\overline{k}}^1 \times \mathbb{P}_{\overline{k}}^1$, then there is nothing to prove. Suppose that this is not the case. By Fact 2.3, we deduce that $\overline{X}$ is isomorphic to the blow-up of $\mathbb{P}_{\overline{k}}^2$ at one $\overline{k}$-rational point and hence $\overline{X}$ contains a single exceptional curve E. By Exercise 2.23, the class of E is fixed by the Galois group and by Lemma 2.24, the exceptional curve E is defined over k and we can therefore blow it down to obtain a del Pezzo surface of degree 9 with a k-rational point. □

Exercise 2.30. Let X be a del Pezzo surface of degree 8 over k. Suppose that $\overline{X}$ is isomorphic to $\mathbb{P}_{\overline{k}}^1 \times \mathbb{P}_{\overline{k}}^1$. If X contains a k-rational point, then X is k-rational.

Theorem 2.31. *Let X be a del Pezzo surface of degree 7. There is a birational morphism $\pi \colon X \to \mathbb{P}_k^2$. The morphism π is the simultaneous blow up of two distinct points of $\mathbb{P}_k^2$ that are k'-rational points, where $k' \supset k$ is an extension of degree at most 2.*

Sketch of proof. The graph of exceptional curves of $\overline{X}$ is the tree with 3 vertices Γ_3. The group $\mathrm{Isom}(X)$ therefore has order two: there is a subgroup of index at most 2 in the Galois group $\mathrm{Gal}(k)$ that acts as the identity on $\mathrm{Pic}\,\overline{X}$. Such a subgroup corresponds to an extension $k' \supset k$ if degree at most 2. The base-change $X \times_k \operatorname{Spec} k'$ is thus isomorphic to the blow up of $\mathbb{P}^2_{k'}$ at two k'-rational points. Argue that the line joining these two points is defined over k. □

We can already start having some expectations on del Pezzo surfaces, from the evidence of the large degree surfaces that we examined:

(1) they sometimes have a k-rational point, regardless of their field of definition;
(2) the existence of a k-rational point seems imply that they are k-rational.

Indeed, we saw that del Pezzo surfaces of degree 7 always have a point. Also, del Pezzo surfaces of degree 8 that are $\overline{k}$ isomorphic to the blow-up of $\mathbb{P}^2_{\overline{k}}$ at two points always have a rational point. We also saw that del Pezzo surfaces of degree 1 necessarily have a rational point: the base point of the anti-canonical linear system. There are two further important results that we state here.

Fact 2.32 (Enriques, Swinnerton-Dyer)**.** Every del Pezzo surface of degree 5 has a k-rational point.

Fact 2.33 (Segre, Manin)**.** Let X be a del Pezzo surface of degree at least 2 over a field k and let p be a k-rational point on X. If the degree of X is 2, then assume that p is not contained either on the ramification locus of the anti-canonical morphism, nor on 4 exceptional curves, defined over $\overline{k}$. Then X is k-unirational.

So far, we only saw examples of k-rationality. We outline a k-unirationality construction in the case of del Pezzo surfaces of degree 3, which is also Segre's original argument for cubic surfaces with a point.

Exercise 2.34. Let X be a smooth cubic surface over a field k and let p be a k-rational point on X. Let $C_p \subset X$ be the intersection of X with the tangent space to X at p.

(1) Show that C_p is geometrically reduced.
(2) If C is geometrically integral, then show that C is k-rational (and singular).
(3) If C is reducible over k, then show that there is a line L contained in C that is defined over k and is k-rational.
(4) If C is irreducible over k and reducible over $\overline{k}$, then show that there are 3 lines through p that are defined over $\overline{k}$ and permuted transitively by $\operatorname{Gal}(k)$.
(5) In Case (2), repeating the construction of the tangent plane starting with the generic point of C shows that X is k-unirational.
(6) In Case (3), the tangent plane to X at a general point q on the line L intersects X along L and a smooth conic Q containing q. Thus, the conic Q is k-rational and the family of such conics parameterized by the points on the line L produces a k-unirational parameterization of X..

The argument in Case (4) is more elaborate. Segre's proof uses that the field is infinite (or at least sufficiently large) to find a suitable different point. A point on a cubic surface contained in 3 lines is called an *Eckardt point*. A cubic surface need not have any Eckardt points, not even over an algebraically closed field. Indeed, in the moduli space of cubic surfaces, the ones containing an Eckardt point is a divisor. Thus, the previous exercise works for a general cubic surface, with no assumption on the position of the point!

There is an alternative unirationality construction for cubic surfaces with a point due to Kollár that works uniformly over every field.