

# EXTENDED ABSTRACT: RATIONAL SIMPLE CONNECTEDNESS AND FANO THREEFOLDS

ANDREA FANELLI

We work over the field of complex numbers.

Fano fibrations naturally appear in the birational classification of algebraic varieties. Let  $X$  be a smooth projective variety with negative Kodaira dimension, then the minimal model program (or MMP) conjecture predicts that  $X$  is birational to a Mori fibration  $\pi: W \rightarrow B$ , which is a “minimal” Fano fibration.

We are interested in studying the existence of rational sections for Fano fibrations. Since smooth Fano varieties are rationally connected [4, 16], the classical result by Graber, Harris and Starr [12] implies that if  $B$  is a curve, then  $\pi$  has a section (i.e. the generic fibre of  $\pi$  has a  $\mathbb{C}(B)$ -point).

The problem becomes much more subtle over a higher dimensional base: it is very easy to construct conic bundles over smooth surfaces having no rational sections.

**Problem 1.** *Can we find a sufficient condition on the fibres of  $\pi$  which guarantees the existence of a rational section, when  $B$  is a smooth surface?*

The first positive answer in this direction is the following classical result.

**Theorem 2** (Tsen-Lang, [17]). *Let  $K = \mathbb{C}(B)$  be the function field of an  $r$ -dimensional variety  $B$  and let  $X_d \subset \mathbb{P}_K^n$  be a degree  $d$  hypersurface over  $K$  which verifies the numerical condition*

$$(1) \quad d^r \leq n.$$

*Then  $X_d(K) \neq \emptyset$  (i.e.  $X_d$  has a  $K$ -point).*

The case  $r = 1$  can be rephrased as follows: for a smooth hypersurface  $X_d \subset \mathbb{P}^n$  over the function field of a curve, the following conditions are equivalent:

- (i) (numerical)  $d \leq n$ ;
- (ii) (geometric)  $X_d$  is rationally connected.

Also, an easy computation shows that (i) is equivalent to the Fano condition.

A very intuitive way to generalise the numerical condition (1) for an arbitrary smooth Fano variety  $X$  consists in asking positivity of some Chern characters.

**Definition 3.** A smooth Fano variety  $X$  is 2-Fano if  $\text{ch}_2(X) > 0$ .

This class of Fano varieties has been introduced in [8] and extensively studied in several papers [6, 2, 3, 20, 22, 1]. Although this condition is (at least in principle) easy to check on examples, it is geometrically very restrictive (see Theorem 7).

In this context, Problem 1 can be rephrased as:

*Is there a geometric analogue of the numerical condition “ $d^2 \leq n$ ” for smooth Fano varieties?*

Inspired by topology, Barry Mazur proposed the following algebro-geometric analogue of simple connectedness in topology: *one asks for rational connectedness of certain moduli spaces of rational curves of sufficiently high degree* ([7, Section 1], [5, Hypothesis 6.8]). This is a *working* definition, since the geometry of moduli spaces of rational curves of high degree on Fano varieties is not well understood (cf. [11, Section 1]).

In order to fix the ideas, we provide a definition of rational simple connectedness for Fano varieties of Picard rank one (see [7] for a more general definition for polarised varieties, [5] for a slightly different notion and [25] for a survey paper on the topic).

Let  $X$  be a smooth Fano variety verifying  $\rho(X) = 1$ . We denote by  $\overline{M}_{0,m}(X, d)$  the coarse moduli space of degree  $d$  stable rational curves with  $m$  marked points on  $X$ .

This moduli space comes with an evaluation map  $\text{ev}_m: \overline{M}_{0,m}(X, d) \rightarrow X^m$ .

**Definition 4** ([7]). The Fano variety  $X$  is *rationally simply connected* if for any  $m \geq 2$  there exists a degree  $d_0(m) > 0$  such that for all  $d \geq d_0(m)$  there exists a canonical component  $M_{d,m}$  of  $\overline{M}_{0,m}(X, d)$  for which the evaluation map

$$\text{ev}_m: M_{d,m} \rightarrow X^m$$

is dominant and its general fibre is rationally connected.

What is surprising about this notion is that the results in [5] provide evidence of the following quite general principle: *the only obstruction to a rational section for a fibration  $\pi: Z \rightarrow B$  over a surface with rationally simply connected fibres lies in some Brauer class of the base  $B$* . Actually, the authors of [5] prove that the vanishing of some Brauer obstruction implies the existence of rational sections when the fibres are rational homogeneous varieties. See also [21] for a discussion on this Brauer obstruction.

Clearly, rationally simply connected varieties are rationally connected, which in turn are simply connected [9, Corollary 4.18]. The core question is the following.

**Question 5.** *Which Fano varieties are rationally simply connected?*

It is surprising how very few classes of examples (and counterexamples) of rationally simply connected Fano varieties are known. It is not too hard to see that the projective space  $\mathbb{P}_{\mathbb{C}}^n$  and the smooth quadric hypersurface  $Q^n \subset \mathbb{P}_{\mathbb{C}}^{n+1}$  are both rationally simply connected (see [11]). Understanding this property for general complete intersection is much trickier and the following result has been obtained.

**Theorem 6** ([7, 10]). *Let  $X_{\underline{d}}^{(n)} \subset \mathbb{P}_{\mathbb{C}}^N$  be a general complete intersection of dimension  $n \geq 3$  and degree  $\underline{d} = (d_1, \dots, d_{N-n})$ . Then*

$$X_{\underline{d}}^{(n)} \text{ is rationally simply connected} \iff \sum_{i=1}^{N-n} d_i^2 \leq N$$

The previous result shows that Tsen-Lang numerical condition (1) is equivalent to rational simple connectedness for general hypersurfaces.

Although the full classification of smooth complex Fano threefolds up to deformation is well-known ([13, 14, 15, 18, 19]) and the families have a very explicit description, we stress that rationally simple connectedness is not a birational property, namely there exist rational Fano varieties which are not rationally simply connected.

The study of rational simple connectedness for smooth Fano threefolds is interesting, also because the notion of 2-Fano is too restrictive in this context, as the following result shows.

**Theorem 7** ([3]). *Let  $X$  be a smooth complex Fano threefold. Then*

$$X \text{ is 2-Fano} \iff X \simeq \mathbb{P}^3 \text{ or } Q^3 \subset \mathbb{P}^4.$$

In [11], the study of rational simple connectedness for smooth Fano threefolds of index two has been started, and we obtain the following result.

**Theorem 8** ([11]). *The smooth quintic Fano threefold  $V_5 \subset \mathbb{P}_{\mathbb{C}}^6$  is rationally simply connected.*

We recall that  $V_5$  is a rational and rigid. The techniques used to prove Theorem 8 involve an explicit description of moduli spaces of curves, looking at the birational geometry of rational Fano varieties (see [23, 24]).

#### REFERENCES

- [1] Carolina Araujo, Roya Beheshti, Ana-Maria Castravet, Kelly Jabbusch, Svetlana Makarova, Enrica Mazzon, Libby Taylor, and Nivedita Viswanathan. Higher fano manifolds. *preprint*, 2021. ([document](#))
- [2] Carolina Araujo and Ana-Maria Castravet. Polarized minimal families of rational curves and higher Fano manifolds. *Amer. J. Math.*, 134(1), 2012. ([document](#))
- [3] Carolina Araujo and Ana-Maria Castravet. Classification of 2-Fano manifolds with high index. In *A celebration of algebraic geometry*, volume 18 of *Clay Math. Proc.* Amer. Math. Soc., Providence, RI, 2013. ([document](#)), 7
- [4] Frédéric Campana. Connexit  rationnelle des vari t s de Fano. *Ann. Sci.  cole Norm. Sup. (4)*, 25(5):539–545, 1992. ([document](#))
- [5] Aise J. de Jong, Xuhua He, and Jason M. Starr. Families of rationally simply connected varieties over surfaces and torsors for semisimple groups. *Publ. Math. Inst. Hautes  tudes Sci.*, (114), 2011. ([document](#))
- [6] Aise J. de Jong and Jason Starr. Higher Fano manifolds and rational surfaces. *Duke Math. J.*, 139(1), 2007. ([document](#))
- [7] Aise J. de Jong and Jason M. Starr. Low degree complete intersections are rationally simply connected. *Preprint*, 2006. ([document](#)), 4, 6
- [8] Aise J. de Jong and Jason M. Starr. A note on fano manifolds whose second chern character is positive. *preprint*, 2006. ([document](#))
- [9] Olivier Debarre. Higher-dimensional varieties: lecture notes for the CIMPA-CIMAT-ICTP School on Moduli of Curves, February 22–March 4, 2016, Guanajuato, M xico. In *Moduli of curves*, volume 21 of *Lect. Notes Unione Mat. Ital.*, pages 49–106. Springer, Cham, 2017. ([document](#))
- [10] Matt DeLand. Relatively very free curves and rational simple connectedness. *J. Reine Angew. Math.*, 699, 2015. 6
- [11] Andrea Fanelli, Laurent Gruson, and Nicolas Perrin. Rational curves on  $V_5$  and rational simple connectedness. *preprint*, 2019. ([document](#)), 8
- [12] Tom Graber, Joe Harris, and Jason Starr. Families of rationally connected varieties. *J. Amer. Math. Soc.*, 16(1), 2003. ([document](#))
- [13] Vasily A. Iskovskih. Fano threefolds. I. *Izv. Akad. Nauk SSSR Ser. Mat.*, 41(3):516–562, 717, 1977. ([document](#))

- [14] Vasily A. Iskovskih. Fano threefolds. II. *Izv. Akad. Nauk SSSR Ser. Mat.*, 42(3):506–549, 1978. ([document](#))
- [15] Vasily A. Iskovskih. Anticanonical models of three-dimensional algebraic varieties. In *Current problems in mathematics, Vol. 12 (Russian)*, pages 59–157, 239 (loose errata). VINITI, Moscow, 1979. ([document](#))
- [16] János Kollár, Yoichi Miyaoka, and Shigefumi Mori. Rationally connected varieties. *J. Algebraic Geom.*, 1(3):429–448, 1992. ([document](#))
- [17] Serge Lang. On quasi algebraic closure. *Ann. of Math. (2)*, 55:373–390, 1952. [2](#)
- [18] Shigefumi Mori and Shigeru Mukai. Classification of Fano 3-folds with  $B_2 \geq 2$ . *Manuscripta Math.*, 36(2):147–162, 1981/82. ([document](#))
- [19] Shigefumi Mori and Shigeru Mukai. Erratum: “Classification of Fano 3-folds with  $B_2 \geq 2$ ” [Manuscripta Math. **36** (1981/82), no. 2, 147–162; MR0641971 (83f:14032)]. *Manuscripta Math.*, 110(3):407, 2003. ([document](#))
- [20] Takahiro Nagaoka. On a sufficient condition for a Fano manifold to be covered by rational  $N$ -folds. *J. Pure Appl. Algebra*, 223(11):4677–4688, 2019. ([document](#))
- [21] Jason M. Starr. *Brauer groups and Galois cohomology of function fields of varieties*. Publicações Matemáticas do IMPA. [IMPA Mathematical Publications]. Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2008. XX Escola de Álgebra. [XX School of Algebra]. ([document](#))
- [22] Taku Suzuki. Higher order minimal families of rational curves and Fano manifolds with nef Chern characters. *J. Math. Soc. Japan*, 73(3):949–964, 2021. ([document](#))
- [23] Hiromichi Takagi and Francesco Zucconi. Geometries of lines and conics on the quintic del Pezzo 3-fold and its application to varieties of power sums. *Michigan Math. J.*, 61(1), 2012. ([document](#))
- [24] Hiromichi Takagi and Francesco Zucconi. The moduli space of genus four even spin curves is rational. *Adv. Math.*, 231(5), 2012. ([document](#))
- [25] Claire Voisin. Sections rationnelles de fibrations sur les surfaces et conjecture de Serre [d’après de Jong, He et Starr]. Number 348, pages Exp. No. 1038, ix, 317–337. 2012. Séminaire Bourbaki: Vol. 2010/2011. Exposés 1027–1042. ([document](#))

UNIV. BORDEAUX, CNRS, BORDEAUX INP, IMB, UMR 5251, F-33400 TALENCE, FRANCE.

*Email address:* `andrea.fanelli@math.u-bordeaux.fr`