EXTENDED ABSTRACT: RATIONAL SIMPLE CONNECTEDNESS AND THE QUINTIC DEL PEZZO THREEFOLD

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The goal of this note is to explain on few examples a generalisation of rational connectedness for smooth Fano varieties, called *rational simple connectedness*. It can be interpreted as an algebro-geometric analogue of simple connectedness in topology, where path-connectedness is replaced by rational connectedness. We work over the field of complex numbers, and we start recalling a classical definition, which was introduced in [9].

Definition 1. A proper variety X is *rationally connected* if for any two general points $x, y \in X$ there exists a morphism $f: \mathbb{P}^1 \to X$ for which $x, y \in f(\mathbb{P}^1)$.

It turns out that, in our setting, rational connectedness is equivalent to the existence of a "very movable" rational curve in X (see [5, Corollary 4.17]).

Proposition 2. *Let X be a smooth variety. Then X is rationally connected if and only if there exists a rational curve $f: \mathbb{P}^1 \to X$ that is* very free*, i.e. such that $f^*T_X \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$ is generated by its global sections.*

In this note we will focus on Fano varieties, i.e. varieties with ample anticanonical bundle. Moreover we assume all Fano varieties to be smooth. These objects play a key role in the birational classfication of algebraic varieties and they are known to be rationally connected (see [1, 9]).

In general topology one shows that a path-connected topological space X is simply connected iff for any pair of points $x, y \in X$ the space of based paths is path connected (in the compact-open topology). So, following an idea by Barry Mazur, an algebro-geometric analogue of simply connectedness (where path-connectedness is replaced by rational connectedness) should require rational connectedness of the space which parametrises rational curves with two base points ([4, Section 1], [3, Hypothesis 6.8]). Unfortunately, this notion gets very technical, since the geometry of moduli spaces of rational curves of high degree on Fano varieties is in general not well understood (cf. [7, Section 1]). Nonetheless, rational simple connectedness has been used in [3] to prove Serre's Conjecture II (see [14] for a survey paper on this topic).

Let X be a Fano variety verifying $\rho(X) = 1$ (a more general definition can be found in [4], see also [3]). We denote by $\overline{\mathrm{M}}_{0,m}(X, d)$ the coarse moduli space of degree d stable rational curves with m marked points on X and let $\mathrm{ev}_m: \overline{\mathrm{M}}_{0,m}(X, d) \to X^m$ be the evaluation map.

Definition 3 ([4])**.** The Fano variety X is *rationally simply connected* if for any $m \geq 2$ there exists a degree $d_0(m) > 0$ such that for all $d \geq d_0(m)$ there exists a canonical component $M_{d,m}$ of $\overline{\mathrm{M}}_{0,m}(X, d)$ for which the evaluation map

$$\mathrm{ev}_m\colon M_{d,m} \to X^m$$

is dominant and its general fibre is rationally connected.

The first example to look at is the projective space $\mathbb{P}^n$: let $\mathrm{Mor}_d^{\mathbf{t}\to\mathbf{x}}(\mathbb{P}^1,\mathbb{P}^n)$ be the variety parametrising degree d morphisms $f\colon\mathbb{P}^1 \to \mathbb{P}^n$ such that $f(t_i) = x_i$ for all $i \in [0,d]$, where $\mathbf{t} = (t_0,\dots,t_d)$ are *distinct* points in $\mathbb{P}^1$ and $\mathbf{x} = (x_0,\dots,x_d)$ are points in $\mathbb{P}^n$. It is not hard to show that $\mathrm{Mor}_d^{\mathbf{t}\to\mathbf{x}}(\mathbb{P}^1,\mathbb{P}^n)$ is rational and deduce that $\mathbb{P}^n$ is rationally simply connected (see [7, Proposition 2.2]).

The second example to study is the quadric hypersurface $Q_n \subset \mathbb{P}^{n+1}$, with $n \geq 3$. As for the previous case, one studies the parameter space $\mathrm{Mor}_d^{\mathbf{t}\to\mathbf{x}}(\mathbb{P}^1,Q_n)$ of degree d morphisms $f\colon\mathbb{P}^1 \to Q_n$ such that $f(t_i) = x_i$ for all $i \in [0,d]$. In this setting, the following holds (see [7, Proposition 2.15]).

(1) If d is odd and $\mathbf{t}$ and $\mathbf{x}$ are general, $\mathrm{Mor}_d^{\mathbf{t}\to\mathbf{x}}(\mathbb{P}^1,Q_n)$ is empty;
(2) if d is even and $\mathbf{t}$ and $\mathbf{x}$ are general, $\mathrm{Mor}_d^{\mathbf{t}\to\mathbf{x}}(\mathbb{P}^1,Q_n)$ is a point.

This easily implies rational connectedness for the general fibre of the evaluation maps when d is even. The case when d is odd requires a finer analysis involving pfaffians to prove that Q_n is rationally simply connected (see [7, Proposition 2.7]).

It is natural to ask which hypersurfaces (or more generally complete intersections) are rationally simply connected.

Theorem 4 ([4, 6])**.** *Let $X_{\underline{d}}^{(n)} \subset \mathbb{P}_{\mathbb{C}}^N$ be a general complete intersection of dimension $n \geq 3$ and degree $\underline{d} = (d_1,\dots d_{N-n})$. Then*

$$X_{\underline{d}}^{(n)} \text{ is rationally simply connected} \iff \sum_{i=1}^{N-n} d_i^2 \leq N$$

The proof of this result is very involved and one of the key ingredient is the existence of a two-dimensional analogue of a very free rational curve.

Definition 5 ([4, 6])**.** Let $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $\pi\colon S \to \mathbb{P}^1$ the first projection. A morphism $f\colon S \to X$ to a smooth variety X is *very twisting* if

(1) f^*T_X is globally generated;
(2) (π, f) is finite and $H^1(S, N_f(-E-2F)) = 0$.

(F is a fibre of π and E is a fibre of the second projection).

Theorem 6. *(*[6, Theorem 1.7]*) A general complete intersection verifying the numerical inequality of Theorem 4 contains a very twisting surface.*

The main result from [7] proves rational simple connectedness for the quintic del Pezzo threefold $V_5 \subset \mathbb{P}^6$, the linear section of the Grassmannian $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ by a general linear subspace $\mathbb{P}^6 \subset \mathbb{P}^9$ (see [8, pag. 60-61], [2, Chapter 7] and [10, Section 5.1] for details).

Theorem 7 ([7])**.** *The quintic del Pezzo threefold $V_5 \subset \mathbb{P}^6$ is rationally simply connected.*

Several results on rationality of moduli spaces of rational curves in low degree on V_5 have been obtained in [12, 13], moreover irreducibility results for these moduli spaces in higher degree appeared in [11, Theorem 7.9].

Our explicit approach to deduce rational simple connectedness of V_5 consists in exploiting an elementary birational map from V_5: the projection from a general line (see [8, pag 147, Example (i)], [2, Section 7.7]). Let l be a line in V_5 and $\phi_l : V_5 \dashrightarrow \mathbb{P}^4$ the projection from l, which is dominant to a quadric threefold Q_3 and is birational (this is a way to show rationality of V_5). The divisor D_l spanned by the lines in V_5 meeting l is contracted by ϕ_l onto a twisted cubic in Q_3. So rational connectedness of spaces parametrising rational curves on V_5 can be reduced to rational connectedness of those parametrising rational curves on the quadric Q_3 with special incidence conditions (see [7, Theorem 4.5]).

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